MAT291 Course Notes

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$\hat{\theta}$

1 Multi-variable Functions

Implicit	Explicit
$F(x, y, z) = 0$	$z = f(x, y)$

A **function** $z = f(x, y)$ assigns to each point (x, y) in a set D a unique real number z in a subset of $\mathbb{R}$. The set D is the **domain** of f. The **range** of f is the set of real numbers z that are assumed as the points (x, y) vary over the domain

1.1 Level/Contour Curves

For a surface $z = f(x, y)$

A **Contour** curve is the path given by setting the surface $z = f(x, y)$ to a constant $z = z_0$.

A **Level** curve is the path given by projecting a **ContourCurve** onto the XY-plane ($z = 0$).

2 Limits

2.1 Two Variable Limit

The function $f(x, y)$ has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{P\to P_0} f(x, y) = L$$

if, given any $\epsilon > 0$, there exists a $\delta < 0$ s.t.

$$|f(x, y) - L| < \epsilon$$

whenever (x, y) is in the domain of f and

$$0 < |PP_0| = \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

2.2 Limit Evaluation Methods

Two approaches taken to determine if a limit exists or does not exist

- Assume the limit exists
  - Factorization
  - Algebraic Conjugate
  - Conjugate and Basic Theorems
- Assume the limit does not exist
  - Use two paths with different results for the limit to show that the limit does not exist

2.3 Interior and Boundary Points

Let R be a region in $\mathbb{R}^2$.

An **Interior Point** P of R lies entirely within R (it is possible to find a disk centered at P with some radius that fits entirely within R).

An **Boundary Point** Q of R lies on the edge of R (every disk centered at Q contains at least one point in R and one point not in R

2.4 Open and Closed Sets

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

$$\{(x, y) : x^2 + y^2 < 9\}$$

is an open region

$$\{(x, y) : x^2 + y^2 \leq 4\}$$

is a closed region

2.5 Two-Path Test for Nonexistence of Limits

If $f(x, y)$ approaches two different values as (x, y) approaches (a, b) along two different paths in the domain of f, then

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

does not exist.

Methods:

- $x = g(u, v), y = h(u, v)$
- $y = mx^n, x = my^n$

3 Continuity

The function f is **continuous** at the point (a,b) provided

- f is defined at (a,b)
- $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
- $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

3.1 Continuity of Composite Functions

IF $u = g(x,y)$ is continuous at (a,b) and $z = f(u)$ is continuous at $g(a,b)$, then the composite function $z = f(g(x,y))$ is continuous at (a,b)

4 Derivatives

4.1 1D Derivative

$$f'(a) = \frac{d}{dx} f(x)|_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

4.2 Multi-variable Partial Derivative

$$f_x(a,b) = \frac{\partial}{\partial x} f(x,y)|_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \frac{\partial}{\partial y} f(x,y)|_{(a,b)} = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

4.3 Clairaut's Theorem

Equality of Mixed Partial Derivatives:

If f_{yx} and f_{xy} are continuous and defined on $D \in \mathbb{R}^2$, then

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y)$$

4.4 Differentiability

The function $z = f(x,y)$ is **differentiable** at (a,b) provided $f_x(a,b)$ and $f_y(a,b)$ exist and the change $\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b)$ equals

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where for fixed a and b, ϵ_1 and ϵ_2 are functions that depend only on Δx and Δy, with $(\epsilon_1, \epsilon_2) \to (0,0)$ as $(\Delta x, \Delta y) \to (0,0)$.

A function is **differentiable** on an open set R if it is differentiable at every point on R.

4.4.1 Conditions for Differentiability

Suppose the function f has

- partial derivatives f_x and f_y on an open set containing (a, b)
- f_x and f_y continuous at (a, b)

Then f is differentiable at (a, b).

4.4.2 Differentiable Implies Continuous

If a function f is differentiable at (a, b), then it is continuous at (a, b)

5 Chain Rule

5.1 One Independent Variable

Let z be a differentiable function of x, y and let x, y be differentiable functions of t. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

5.2 Two Independent Variables

Let z be a differentiable function of x, y and let x, y be differentiable functions of s **and** t. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

5.3 Implicit Differentiation

Let F be differentiable on its domain and suppose $F(x, y) = 0$ defines y as a differentiable function of x. Provided $F_y \neq 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

6 Directional Derivatives and Gradient

6.1 Directional Derivative

Let f be differentiable at (a, b) and let $u = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The **Directional Derivative** of f at (a, b) in the direction of u is

$$D_u f(a, b, c) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2, c + hu_3) - f(a, b, c)}{h}$$

$$D_u f(a, b, c) = \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle \cdot \langle u_1, u_2, u_3 \rangle$$

$$D_u f(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}$$

6.2 Gradient

Let f be differentiable at the point (x, y). The *gradient* of f at (x, y) is the vector valued function

$$\nabla f(x,y) = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial z}{\partial z} \right\rangle = \left\langle f_x, f_y, f_z \right\rangle$$

6.3 Directions of Change

Let f be differentiable at (a, b) with $\nabla f(a,b,c) \neq 0$

- The maximum rate of increase of f is in the $\nabla f(a,b,c)$ direction
  - The rate of change in this direction is $|\nabla f(a,b,c)|$
- The minimum rate of increase of f is in the $-\nabla f(a,b,c)$ direction
  - The rate of change in this direction is $-|\nabla f(a,b,c)|$
- The directional derivative is zero in any direction orthogonal to $\nabla f(a,b,c)$

6.4 Gradient and Level Curves

Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a,b)$, provided $\nabla f(a,b) \neq 0$

The tangent at any point on a level curve contour is orthogonal to the gradient.

7 Multi-variable Taylor Series

7.1 Tangent Planes

The Tangent Plane to a function f at a point (a, b, c) is given by

$$\nabla F(x,y,z) \cdot \langle x-a, y-b, z-c \rangle = 0$$

7.2 Linear Approximations

$$L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

$$L(x,y) = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) + f(a,b,c)$$

7.3 Sensitivity Analysis

$$z - c = \frac{\partial f}{\partial x}|_{(a,b)}(x-a) + \frac{\partial f}{\partial y}|_{(a,b)}(y-b)$$

And let

$$\Delta z = z - c \quad \Delta x = x - a \quad \Delta y = y - b$$

So

$$\Delta z = \frac{\partial f}{\partial x}|_{(a,b)}\Delta x + \frac{\partial f}{\partial y}|_{(a,b)}\Delta y$$

The per-unit form of Sensitivity Analysis is given by

$$\frac{dz}{z} = \frac{\partial f}{\partial x}|_{(a,b)}\frac{dx}{x}\frac{x}{z} + \frac{\partial f}{\partial y}|_{(a,b)}\frac{dy}{y}\frac{y}{z}$$

Substituting

$$\frac{dz}{z} = \frac{\partial f}{\partial x}|_{(a,b)}\frac{dx}{x}\frac{a}{c} + \frac{\partial f}{\partial y}|_{(a,b)}\frac{dy}{y}\frac{b}{c}$$

7.4 Small Signal Modelling

7.4.1 State Space Equations

$$\frac{dx_1}{dt} = a_{11}x_1^2 + a_{12}x_2 + b_{11}u_1 + b_{12}u_2$$
$$\frac{dx_2}{dt} = a_{21}x_1^2 + a_{22}x_2 + b_{21}u_1 + b_{22}u_2$$

7.4.2 Output Equations

$$y_1 = c_{11}x_1 + c_{12}x_2 + d_{11}u_1 + d_{12}u_2$$
$$y_2 = c_{21}x_1 + c_{22}x_2 + d_{21}u_1 + d_{22}u_2$$

7.4.3 State Variable Equilibrium Condition

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(x_{10}, x_{20}, U_1, U_2) \\ f_2(x_{10}, x_{20}, U_1, U_2) \end{bmatrix}$$

7.4.4 Small Signal Model in Matrix Form

$$\begin{bmatrix} \frac{d\hat{x_1}}{dt} \\ \frac{d\hat{x_2}}{dt} \end{bmatrix} = J_f \begin{bmatrix} \hat{x_1} \\ \hat{x_2} \end{bmatrix} + B_f \begin{bmatrix} \hat{u_1}(t) \\ \hat{u_2}(t) \end{bmatrix}$$

Where J_f and B_f are Jacobian Matrices

7.4.5 Output Equilibrium Point

$$\begin{bmatrix} g_1(x_{10}, x_{20}, U_1, U_2) \\ g_2(x_{10}, x_{20}, U_1, U_2) \end{bmatrix} = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix}$$

7.4.6 Output Small Signal Equation

$$\begin{bmatrix} \hat{y_1} \\ \hat{y_2} \end{bmatrix} = \mathbf{J_g} \begin{bmatrix} \hat{x_1} \\ \hat{x_2} \end{bmatrix} + \mathbf{B_g} \begin{bmatrix} \hat{u_1}(t) \\ \hat{u_2}(t) \end{bmatrix}$$

Where J_g and B_g are Jacobian Matrices

8 Double Integrals

8.1 Double Integrals over Rectangular Regions

A function f defined on a rectangular region R in the xy-plane is **integrable** on R if

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$$

exists for all partitions of R and for all choices of (x_k^*, y_k^*) within those partitions. The double integral of f over R is

$$\iint_R f(x,y) dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*) \Delta A_k$$

8.2 Iterated Integrals and Fubini's Theorem

Let f be continuous on the **rectangular** region $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of two iterated integrals:

$$\iint_R f(x,y) dA = \int_c^d \int_a^b f(x,y) dx dy = \int_a^b \int_c^d f(x,y) dy dx$$

8.3 Average Value of a Function over Rectangular Region

The average value of an integrable function f over a region R is

$$\bar{f} = \frac{1}{\text{Area of} R} \iint_R f(x,y) dA$$

8.4 Double Integrals over General Regions

Let R be a region bounded below and above by the graphs of the continuous functions $y = g(x)$ and $y = h(x)$, respectively, and by the lines $x = a$ and $x = b$. If f is continuous on R, then

$$\iint_R f(x,y) dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y) dy dx$$

Let R be a region bounded on the left and right by the graphs of the continuous functions $x = g(y)$ and $x = h(y)$, respectively, and the lines $y = c$ and $y = d$. If f is continuous on R, then

$$\iint_R f(x,y) dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y) dx dy$$

8.5 Double Integrals over Polar Regions

Let f be continuous on the region R in the xy plane expressed in polar coordinates as $R = \{(r,\theta) : 0 \le a \le r \le b, \alpha \le \theta \le \beta\}$, where $\beta - \alpha \le 2\pi$.

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))rdrd\theta$$

For $R = \{(r,\theta) : 0 \le g(\theta) \le r \le h(\theta), \alpha \le \theta \le \beta\}$

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos(\theta), r\sin(\theta))rdrd\theta$$

8.6 Area of Polar Regions

$$A = \iint_R dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} rdrd\theta$$

8.7

9 Triple Integrals

9.1 Triple Integrals in Rectangular Coordinates

$$\iiint_D f(x,y,z)dV = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x,y,z)dzdydx$$

9.2 Average Value of a Function of Three Variables

$$\bar{f} = \frac{1}{\text{Volume of} D} \iiint_D f(x,y,z)dV$$

9.3 Triple Integrals in Cylindrical Coordinates

$$\iiint_D f(x,y,z)dV = \int_a^b \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)rdzdrd\theta$$

9.4 Change of Variables for Common Coordinate Systems

Coordinates	Variables						
	x	y	z	r	θ	ρ	ϕ
Cartesian	x	y	z	$\sqrt{x^2+y^2}$	$\tan^{-1}(\frac{y}{x})$	$\sqrt{x^2+y^2+z^2}$	$\cos^{-1}(\frac{z}{\rho})$
Cylindrical	$r\cos(\theta)$	$r\sin(\theta)$	z	r	θ	$r\csc(\theta)$	$\cos^{-1}(\frac{z}{\rho})$
Spherical	$\rho\sin(\phi)\cos(\theta)$	$\rho\sin(\phi)\sin(\theta)$	$\rho\cos(\phi)$	$\rho\sin(\phi)$	θ	ρ	ϕ

10 Change of Variables

10.1 Jacobian Determinant/Matrix

Given a transformation $T : x = g(u, v), y = h(u, v)$, where g and h are differentiable on a region of the uv-plane, the **Jacobian Determinant** is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial(x)}{\partial(u)} & \frac{\partial(x)}{\partial(v)} \\ \frac{\partial(y)}{\partial(u)} & \frac{\partial(y)}{\partial(v)} \end{vmatrix} = \frac{\partial(x)}{\partial(u)}\frac{\partial(y)}{\partial(v)} - \frac{\partial(x)}{\partial(v)}\frac{\partial(y)}{\partial(u)}$$

Given a transformation $T : x = g(u, v), y = h(u, v), z = p(u, v, w)$, where g,h and p are differentiable on a region of the uv-plane, the **Jacobian Determinant** is

$$J(u,v) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial(x)}{\partial(u)} & \frac{\partial(x)}{\partial(v)} & \frac{\partial(x)}{\partial(w)} \\ \frac{\partial(y)}{\partial(u)} & \frac{\partial(y)}{\partial(v)} & \frac{\partial(y)}{\partial(w)} \\ \frac{\partial(z)}{\partial(u)} & \frac{\partial(z)}{\partial(v)} & \frac{\partial(z)}{\partial(w)} \end{vmatrix}$$

10.2 Change of Variables Integrals

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),h(u,v))|J(u,v)|dA$$

$$\iiint_D f(x,y,z)dV = \iint_S f(g(u,v,w),h(u,v,w),p(u,v,w))|J(u,v,w)|dV$$

11 Surface Integrals

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $r(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, where u and v vary over $R = \{(u, v) : a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors

$$t_u = \frac{\partial x}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$$t_v = \frac{\partial x}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

are continuous on R and the normal vector $t_u \times t_v$ is nonzero on R. Then the surface integral of f over S is

$$\iint_S f(x,y,z)dS = \iint_R f(x(u,v),y(u,v),z(u,v)|t_u \times t_v|dA$$

11.1 Surface Area

$$\text{Surface Area} = \iint_S 1dS = \iint_R 1|t_u \times t_v|dA$$

12 Curl and Circulation

$$\text{Circ} = \oint_C F \cdot T ds$$

$$\text{Curl} = \nabla \times F = \lim_{A \to 0} \frac{\oint_C F \cdot T ds}{A}$$

where A is the area enclosed by contour C

$$\text{Curl} = \nabla \times F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \left\langle f(x, y, z), g(x, y, z), h(x, y, z) \right\rangle$$

13 Divergence and Flux

$$\text{Flux} = \oint_C F \cdot n ds$$

$$\text{Div} = \nabla \cdot F = \lim_{A \to 0} \frac{\oint_C F \cdot n ds}{A}$$

where A is the area enclosed by contour C

$$\text{Div} = \nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle f(x, y, z), g(x, y, z), h(x, y, z) \right\rangle$$

14 Vector Identities

14.1 Dot Product

$$A \cdot B = \langle A_1, A_2, A_3 \rangle \cdot \langle B_1, B_2, B_3 \rangle = A_1 B_1 + A_2 B_2 + A_3 B_3$$

14.2 Cross Product

$$A \times B = \langle A_1, A_2, A_3 \rangle \times \langle B_1, B_2, B_3 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

14.3 Scalar Triple Product

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

14.4 Divergence/Curl Linearity

$$\nabla \cdot (A + B) = \nabla \cdot A + \nabla \cdot B$$

$$\nabla \cdot (A + B) = \nabla \times A + \nabla \times B$$

14.5 Second Derivatives

14.5.1 Source Free Field

$$\nabla \cdot (\nabla \times A) = 0$$

14.5.2 Rotation Free Field

$$\nabla \times (\nabla \Psi) = 0$$

14.5.3 Scalar Laplacian

$$\nabla \cdot (\nabla \Psi) = \nabla^2 \Psi$$

14.5.4 Vector Laplacian

$$\nabla(\nabla \cdot A) - \nabla \times (\nabla \times A) = \nabla^2 A$$

15 Stokes Theorem

$$\text{circ}(F) = \oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot ndS$$

16 Divergence Theorem

$$\text{flux}(F) = \iint_S F \cdot nds = \iiint_D (\nabla \cdot F) dV$$

17 Useful Geometries

17.1 Normal Vectors

Normal for a sphere with equation $x^2 + y^2 + z^2 = \rho^2$:

$$< \frac{x}{z}, \frac{y}{z}, 1 >$$

Surface	Explicit		
	Equation	Normal Vector n	Magnitude
Cylinder	$x^2 + y^2 = a^2$	$< x, y, 0 >$	a
Cone	$x^2 + y^2 = z^2$	$< \frac{x}{z}, \frac{y}{z}, -1 >$	$\sqrt{2}$
Sphere	$x^2 + y^2 + z^2 = a^2$	$< \frac{x}{z}, \frac{y}{z}, 1 >$	$\frac{a}{z}$
Paraboloid	$x^2 + y^2 = z$	$< 2x, 2y, -1 >$	$\sqrt{1 + 4(z^2 + y^2)}$

Surface	Parametric		
	Equation	Normal Vector $n = t_u \times t_v$	Magnitude $\|t_u \times t_v\|$
Cylinder	$r =< a\cos(u), a\sin(u), v >$	$< a\cos(u), a\sin(u), 0 >$	a
Cone	$r =< v\cos(u), v\sin(u), v >$	$< v\cos(u), v\sin(u), -v >$	$\sqrt{2}v$
Sphere	$r =< a\cos(u)\sin(v), a\sin(u)\sin(v), a\cos(v) >$	$r =< a^2\cos(u)\sin^2(v), a^2\sin(u)\sin^2(v), a^2\sin(v)\cos(v) >$	$a^2\sin(v)$
Paraboloid	$r =< v\cos(u), v\sin(u), v^2 >$	$< 2v^2\cos(u), 2v^2\sin(u), -v >$	$v\sqrt{1+4v^2}$

18 Dirac Delta Distribution

$$\partial(x-a) = \begin{cases} 0 & \text{for } x \neq a \\ \infty & \text{for } x = a \end{cases}$$

18.1 Dirac Delta Integrals

Area under the distribution is 1:

$$\int_{-\infty}^{\infty} \partial(x-a)dx = 1$$

$$\int_{-\infty}^{a} \partial(x-a)dx = \int_{a}^{\infty} \partial(x-a)dx = \frac{1}{2}$$

Spherical Coordinate Generalization:

$$\int_{0}^{r>0} \partial(r)dr = \frac{1}{2}$$

Sampling/Shifting Property:

$$\int_{-\infty}^{\infty} f(x)\partial(x-a)dx = f(a)$$

19 Scalar Density Dirac Distributions

Shape	Cartesian	Spherical	Cylindrical
Point	$\rho = Q\delta(x)\delta(y)\delta(z)$	$\rho = \frac{Q\delta(r)}{2\pi r^2}$	
Infinite Line	$\rho = \lambda\delta(x)\delta(y)$		$\rho = \frac{\lambda\delta(r)}{\pi r}$
Infinite Plane	$\rho = \sigma\delta(z)$		
Infinite Cylinder			$\rho = \sigma\delta(r-R)$
Sphere		$\rho = \sigma\delta(r-R)$	

20 Vector Valued Flux Density Dirac Distributions

Shape	Cartesian	Spherical	Cylindrical
Planar Sheet	$\vec{J} = J_s\delta(z)\hat{x}$		
Line	$\vec{J} = I\delta(x)\delta(y)\hat{z}$		$\vec{J} = \frac{I\delta(r)}{\pi r}\hat{z}$
Cylinder (Axial)			$\vec{J} = J_s\delta(r-R)\hat{z}$
Cylinder (Circumferential)	$\vec{J} = J_s[\delta(y+R) - \delta(y-R)]\hat{x}$		$\vec{J} = J_s\delta(r-R)\hat{\theta}$

21 Divergence Theorem LHS Surface Integrals

Shape	Divergence Theorem LHS
Cylinder	$2\pi rLf(r)$
Line	$2\pi rLf(r)$
Plane	$2Af(z)$, A = plane area
Sphere	$4\pi r^2 f(r)$

22 Stokes Theorem LHS Surface Integrals

Shape	Stoke's Theorem LHS
Solenoidal (Axial, $\hat{z}$)	$zf(z)$
Solenoidal (Circumferential, $\hat{\theta}$)	$2\pi rf(r)$
Toroidal (Donut)	$2\pi rf(r)$
Plane ($F \parallel \hat{x}$,$J \parallel \hat{z}$)	$2xf(y)$

23 Green's Circulation Theorem

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $F = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C F \cdot dr = \oint_C f dx + g dy = \iint_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dA$$

24 Green's Flux Theorem

Let C be a simple closed piecewise-smooth curve, oriented counterclockwise, that encloses a connected and simply connected region R in the plane. Assume $F = \langle f, g \rangle$, where f and g have continuous first partial derivatives in R. Then

$$\oint_C F \cdot dr = \oint_C f dy - g dx = \iint_R \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} dA$$

25 Trigonometry

25.1 Trig Identities

25.1.1 Half Angle Identities

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

25.1.2 Double Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\cos(2x) = 2\cos^2(x) - 1$$

$$\cos(2x) = 1 - 2\sin^2(x)$$

25.2 Hyperbolic Trig

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

25.3 Hyperbolic Trig Identities

$$\sinh(-x) = -\sinh(x)$$
$$\cosh(-x) = \cosh(x)$$
$$\cosh^2(x) - \sinh^2(x) = 1$$
$$1 - \tanh^2(x) = \operatorname{sech}^2(x)$$
$$\sinh(x+y) = \sinh(x)\cosh(x) + \cosh(y)\sinh(y)$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(y)\sinh(y)$$