MAT290 Cheat Sheet

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1 Complex Numbers

$$z = a + ib$$
$$i = \sqrt{-1}$$

1.1 Modulus and Argument

The **modulus** or **absolute value** of $z + iy$ is

$$|z| = \sqrt{x^2 + y^2}$$

The **argument** of a complex number must satisfy the equation

$$\tan(\theta) = \frac{y}{x} \implies \theta = \tan^{-1}(\frac{y}{x})$$

1.2 Polar Form

$$z = x + iy = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$$

where r is the **modulus** of z and θ is the **argument** of z

1.3 Complex Conjugate

Complex Number	Complex Conjugate
$z = x + iy$	$\bar{z} = x - iy$
$z = re^{i\theta}$	$\bar{z} = re^{-i\theta}$

1.4 Powers of z

For $n \in \mathbb{R}$

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

For $\alpha \in \mathbb{C}$

$$z^\alpha = e^{\alpha \ln z}$$

1.5 DeMoivre's Formula

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

1.6 Exponential Function

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

$$|e^z| = e^x \quad \arg e^z = y + 2k\pi$$

1.7 Complex Logarithms

For $z \neq 0$ and $\theta = \arg z$

$$\ln(z) = \log_e |z| + i(\theta + 2n\pi), \quad n = 0, \pm 1, \pm 2, ...$$

1.8 Principal Logarithm

$$\operatorname{Ln} z = \log_e |z| + i(\operatorname{Arg} z)$$

Where Arg is the **principal argument**

1.9 Complex Powers

If z and α are both complex powers

$$z^\alpha = e^{\alpha \ln(z)}$$

1.10 Trigonometric Functions

For any complex number $z = x + iy$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\frac{d}{dz}\sin(z) = \cos(z) \quad \frac{d}{dz}\cos(z) = -\sin(z)$$

1.11 Trig Identities

$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\csc(x) = \frac{1}{\sin(x)}$
$\sec(x) = \frac{1}{\cos(x)}$	$\cot(x) = \frac{\cos(x)}{\sin(x)}$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$
$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$$
$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$$
$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

1.12 Trig Integral Identities

1.13 Hyperbolic Trig Functions

For any complex number $z = x + iy$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\frac{d}{dz}\sinh(z) = \cosh(z) \qquad \frac{d}{dz}\cosh(z) = \sinh(z)$$

1.14 Hyperbolic Trig Identities

$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$	$\operatorname{csch}(z) = \frac{1}{\sinh(z)}$
$\operatorname{sech}(z) = \frac{1}{\cosh(z)}$	$\coth(z) = \frac{\cosh(z)}{\sinh(z)}$

$$\sinh(-z) = -\sinh(z)$$
$$\cosh(-z) = \cosh(z)$$
$$\cosh^2(z) - \sinh^2(z) = 1$$
$$1 - \tanh^2(z) = \operatorname{sech}^2(z)$$
$$\sinh(x+y) = \sinh(x)\cosh(x) + \cosh(y)\sinh(y)$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(y)\sinh(y)$$
$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$
$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

2 Differential Equations

2.1 Autonomous First-Order DE's

$$\frac{dy}{dx} = f(y)$$

2.2 Homogenous DE's

$$... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

2.3 Separable Equations

$$\frac{dy}{dx} = g(x)h(y)$$

2.4 Linear First-Order DE's

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

The solution to a linear differential equation is of the form

$$y = y_c + y_p$$

where y_c is the solution to the associated homogeneous equation and y_p is the particular solution of the non-homogeneous equation.

2.4.1 Standard Form

Divide both sides of Linear DE equation by a_1:

$$\frac{dy}{dx} + P(x)y = f(x)$$

2.4.2 Integrating Factor

To solve a Linear DE, first find its **standard form**

$$\frac{dy}{dx} + P(x)y = f(x)$$

and then multiply by the **integrating factor**

$$e^{\int P(x)dx}$$

to get

$$\frac{dy}{dx}e^{\int P(x)dx} + P(x)e^{\int P(x)dx}y = f(x)e^{\int P(x)dx}$$

$$\frac{dy}{dx}[e^{\int P(x)dx}y] = f(x)e^{\int P(x)dx}$$

2.5 Existence and Uniqueness Theorem

Suppose $a_n(x), a_{n-1}(x), ..., a_1(x), a_0(x)$, and $g(x)$ are continuous in some interval I containing x_0 and $a_n(x) \neq 0$ in I. Then the IVP has a solution in I that is unique

2.6 Superposition Principle for Homogeneous Equations

Let $y_1, y_2, ..., y_k$ be solutions of an nth order homogeneous differential equation on an interval I. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_k y_k(x)$$

is also a solution to the differential equation on I

2.7 Linear Dependence/Independence

A set of functions $f_1(x), f_2(x), ..., f_n(x)$ is said to be linearly dependent on an interval I if there exists constants $c_1, c_2, ..., c_n$ not all zero, such that

$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be $linearlyindependent$

2.8 Wronskian

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ ... & ... & & ... \\ f_1^{(n-1)} & f_2^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix}$$

2.9 Criterion for Linearly Independent Solutions

Let $y_1, y_2, ..., y_n$ be n solutions of the homogeneous linear nth order differential equation. The set of solutions is linearly independent on I iff $W(y1, y2, ..., y_n) \neq 0$ for every x in the interval

2.10 General Solution for Homogeneous Equations

$$y = y_c = c_1 y_1(x) + c_2 y_2(x) + ... + c_k y_k(x)$$

2.11 General Solution for Non-homogeneous Equations

$$y = y_c + y_p = c_1 y_1(x) + c_2 y_2(x) + ... + c_k y_k(x) + y_p$$

2.12 Reduction of Order

For an ODE of the form

$$y'' + P(x)y' + Q(x)y = 0$$

If $y_1(x)$ is a known solution of the ODE on I and $y_1(x) \neq 0$ for every x in I, then

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

2.13 Undetermined Coefficients

2.14 Variation of Parameters

$$y_{nh}(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$u_1' = \frac{W_1}{W} = \frac{-y_2 f(x)}{W} \quad u_2' = \frac{W2}{W} = \frac{y_1 f(x)}{W}$$

$$u_1 = \int_0^x \frac{-y_2(t)f(t)}{W}dt \quad u_2 = \int_0^x \frac{y_1(t)f(t)}{W}dt$$

$$y_{nh}(x) = y_1(x)\int_0^x \frac{-y_2(t)f(t)}{W}dt + y_2(x)\int_0^x \frac{y_1(t)f(t)}{W}dt$$

2.15 Solution Types for Homogenous 2nd Degree Linear ODE's

For an ODE of the form:

$$ay'' + by' + c = 0$$

and characteristic/auxiliary polynomial:

$$ar^2 + br + c = 0$$

with roots

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

2.15.1 Distinct Roots - Overdamped

$$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

2.15.2 Repeated Roots - Critically Damped

e.g. $r_1 = r_2 = r$

$$y_h = c_1 e^{rt} + c_2 t e^{rt}$$

2.15.3 Complex Conjugate Roots - Underdamped

If $r1, r2$ complex, then we can write $r1 = a + ib, r2 = a - ib$

$$y_h = e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$$

3 Laplace Transform

Let f be a function defined for $t \geq 0$

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

3.1 Exponential Order

A function is of **exponential order** if there exists constants c, $M > 0$, and $T > 0$ s.t. $|f(t)| \leq Me^{ct}$ for all $t > T$

If $f(t)$ piecewise continuous on the interval $[0, \infty]$ and of exponential order, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

3.2 Basic Laplace Transforms

$\mathscr{L}\{1\} = \frac{1}{s}$	$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$
$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$	$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$
$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$	$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$

3.3 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at} f(t)\}(s) = \mathscr{L}\{f(t)\}(s-a) = F(s-a)$$

3.4 Second Translation Theorem

The **unit step function** $\mathrm{u}(t-a)$ is defined as

$$\mathrm{u}(t-a) = \begin{cases} 0 & \text{if } 0 \leq t < a \\ 1 & \text{if } t \geq a \end{cases}$$

If $\mathscr{L}\{f(t)\} = F(s)$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathrm{u}(t-a)\} = e^{-as} F(s)$$

3.5 Transforms of Derivatives

If $f, f', ...f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order, and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - f^{(n-1)}(0)$$

where $F(s) = \mathscr{L}\{f(t)\}$

$$\mathscr{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

3.6 Derivatives of Transforms

If $\mathscr{L}\{f(t)\} = F(s)$ and $n = 1, 2, 3, ...$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

3.7 Transform of Integrals

$$\mathscr{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$$

$$\int_0^t f(\tau)d\tau = \mathscr{L}^{-1}\{\frac{F(s)}{s}\}$$

3.8 Convolution

3.8.1 Convolution Operation

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$, then the **convolution** of f and g, denoted by the symbol $f * g$, is

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$$

3.8.2 Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)G(s)$$

3.8.3 Convolution in Inverse

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g$$

3.9 Dirac Delta Function

3.9.1 Unit Impulse

$$\delta_a(t-t_0) = \begin{cases} 0 & \text{if } 0 \le t < t_0 - a \\ \frac{1}{2a} & \text{if } t_0 + a \le t \le t_0 + a \\ 0 & \text{if } t \ge t_0 + a \end{cases}$$

3.9.2 Dirac Delta Definition

$$\delta(t-t_0) = \begin{cases} \infty & \text{if } t = t_0 \\ 0 & \text{if } t \ne t_0 \end{cases}$$

Area under the distribution is 1:

$$\int_{-\infty}^{\infty} \delta(x-a)dx = 1$$

Sampling/Shifting Property:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

3.9.3 Dirac Delta Transform

For $t_0 > 0$

$$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$$
$$\mathscr{L}\{\delta(t)\} = 1$$

3.10 Additional Laplace Properties

3.10.1 Transform of a Periodic Function

If $f(t)$ is

- is piecewise continuous on $[0, \infty)$
- of exponential order
- periodic with period T

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

3.10.2 Volterra Integral Equation

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)d\tau$$

Taking Laplace transform of both sides:

$$F(s) = G(s) + \mathscr{L}\{f(t) * h(t)\} = G(s) + F(s)H(s)$$

3.10.3 Integrodifferential Equation

Circuit Analogy:

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$

4 Inverse Laplace Transforms

4.1 Basic Inverse Laplace Transforms

$\mathscr{L}^{-1}\{\frac{1}{s}\} = 1$	$\mathscr{L}^{-1}\{1\} = \delta(t)$
$\mathscr{L}\{\frac{1}{s-a}\} = e^{at}$	$\mathscr{L}^{-1}\{\frac{n!}{s^{n+1}}\} = t^n$
$\mathscr{L}^{-1}\{\frac{k}{s^2+k^2}\} = \sin(kt)$	$\mathscr{L}^{-1}\{\frac{s}{s^2+k^2}\} = \cos(kt)$
$\mathscr{L}^{-1}\{\frac{k}{s^2-k^2}\} = \sinh(kt)$	$\mathscr{L}^{-1}\{\frac{s}{s^2-k^2}\} = \cosh(kt)$

4.2 The Inverse Laplace Transform

Suppose that $f(t)$ is a continuous and of exponential order $|f(t)| \leq ce^{\alpha t}$. Then, the Inverse Laplace Transform is given by

$$f(t) = \mathscr{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(z)e^{zt}dt$$

Provided $\sigma > \alpha$

4.3 Inverse Laplace Transform Integral

For some t, as $x \to -\infty$ and $|y| \to \infty$

$$f(t) = \mathscr{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(z)e^{zt}dt = \sum_{k=1}^{n} \text{Res}(e^{st}F(s), s_k)$$

For some t, as $x \to +\infty$ and $|y| \to \infty$

$$f(t) = \mathscr{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(z)e^{zt}dt = 0$$

5 Complex Analysis

5.1 Sets

5.1.1 Neighborhood

A circle with radius p and center at z_0 is given by

$$|z - z_0| = p$$

Note that $|z - z_0|$ is equivalently the distance between two points.
This circular region (open disk) is called a **neighborhood**.

5.1.2 Interior Point

A point z_0 is called an **Interior Point** of a set S if there exists some neighborhood of z_0 that lies entirely within S.

5.1.3 Boundary Point

A point z_0 is called an **Boundary Point** of a set S if any neighborhood of z_0 contains at least one point in S and one point not in S.

5.1.4 Open Sets

If every point z in S is an interior point, then S is an **open** set.

5.1.5 Closed Sets

If S contains all its boundary points, then S is a **closed** set.

5.1.6 Connected Set

A set is connected if any two points z_1, z_2 in S can be connected by a polygonal line that lies entirely in the set.

5.1.7 Domain

A set S which is open and connected is called a **domain**.

5.2 Functions

The image w of a complex number will be some other complex number $u + iv$

$$w = f(z) = u(x, y) + iv(x, y)$$

where $f(z)$ is a complex function

5.3 Stream Flow

A complex function can also be interpreted as a **Two-Dimensional Fluid Flow**

$$f(z) = u(x, y) + iv(x, y)$$

$$\frac{dx}{dt} = u(x, y) \quad \frac{dy}{dt} = v(x, y)$$

5.4 Limit of a Function

Suppose the function f is defined in some neighborhood of z_0, except possibly at z_0 itself. Then f is said to possess a limit at z_0, written

$$\lim_{z \to z_0} f(z) = L$$

if for each $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_0| < \delta$

5.5 Continuity

A function is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

5.6 Derivative

Suppose the complex function f is defined in a neighborhood of a point z_0. The derivative of f at z_0 is

$$f'(z_0) = \lim_{\Delta \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Similar to real functions, **differentiability implies continuity**

5.7 Analyticity

A complex function $w = f(z)$ is said to be analytic at a point z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0

5.7.1 Entire Function

Functions that are analytic at any z are called **entire** functions

5.8 Cauchy-Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy-Riemann Equations**

$$\frac{du}{dx} = \frac{dv}{dy} \quad \frac{du}{dy} = -\frac{dv}{dx}$$

5.9 Condition for Analyticity

Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

5.10 Harmonic Functions

A real-value function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

$$\nabla^2 \phi(x, y) = \frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} = 0$$

Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then the functions $u(x, y)$ and $v(x, y)$ are **harmonic** functions

6 Complex Integration

Let f be defined at points of a smooth curve C defined by $x = x(t)$,$y = y(t)$,$a \leq t \leq b$. The contour integral of f along C is

$$\int_C f(z)dz = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k$$

6.1 Contour Integrals

If f continuous on a smooth curve C given by $z(t) = x(t) + iy(t), a \leq t \leq b$, then

$$\int_C f(z)dz = \int_a^b f(r(t))r'(t)dt = \int_a^b f(r)dr$$

6.2 Bounding Theorem

If f is continuous on a smooth curve C and if $|f(z)| \leq M$ for all z on C, $|\int_C f(z)dz| \leq ML$, where L is the length of C.

6.3 Circulation and Net Flux

6.3.1 Book Definition

For $f(z) = u(x, y) + i(x, y)$ and $\vec{F} = \langle U, V \rangle$

$$\oint_C \overline{f(z)}dz = \oint_C (u - iv)(dx + idy) = \Big(\oint_C f \cdot Tds\Big) + i\Big(\oint_C f \cdot Nds\Big)$$

$$\text{circ} = \text{Re}\Big(\oint_C \overline{f(z)}dz\Big) = \Big(\oint_C f \cdot Tds\Big)$$

$$\text{flux} = \text{Im}\Big(\oint_C \overline{f(z)}dz\Big) = \Big(\oint_C f \cdot Nds\Big)$$

where T and N are the unit tangent and unit normal vectors to the positively oriented simple closed contour C

6.3.2 Alternative (Nachman Definition)

For $f(z) = u(x, y) + iv(x, y)$ and a vector field with $-V$ component $\vec{F} = \langle U, -V \rangle = \langle P, Q \rangle$

$$\oint_C f(z)dz = [\text{circ}(\vec{F})] + i[\text{flux}(\vec{F})]$$

$$\text{circ} = \text{Re}\Big(\oint_C f(z)dz\Big) = \Big(\oint_C f \cdot Tds\Big) = \int_C Pdx + Qdy$$

$$\text{flux} = \text{Im}\Big(\oint_C f(z)dz\Big) = \Big(\oint_C f \cdot Nds\Big) = \int_C Pdy - Qdx$$

6.4 Cauchy Goursat

Suppose a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D

$$\oint_C f(z)dz = 0$$

Suppose $C, C_1, ..., C_n$ are simple closed curves with a positive orientation such that $C, C_1, ..., C_n$ are interior to C but the regions interior to each C_k, $k = 1, 2, ..., n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, ..., n$, then

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$$

Additionally, for any $z_0 \in \mathbb{C}$ interior to any simple closed contour C, then

$$\oint_C \frac{1}{(z-z_0)^n} dz = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{if } n \in \mathbb{Z}, n \neq 1 \end{cases}$$

6.5 Independence of Path and Implications of Analyticity

Let z_0 and z_1 be points in domain D. A contour integral $\int_C f(z)dz$ is **independent of the path** if the value of the integral is the same for any contour C in C with initial and end points z_0 and z_1.

If f is an *analytic* in a simply connected domain D, then $\int_C f(z)dz$ is independent of the path C.

6.6 Existence of an Antiderivative

If f is analytic in a simply connected domain D, then f has an anti-derivative in D; that is, there exists a function F s.t. $F'(z) = f(z)$ for all z in D

6.7 Fundamental Theorem for Contour Integrals

Suppose f is continuous in a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point z_0 and terminal point z_1,

$$\int_C f(z)dz = F(z_1) - F(z_0)$$

6.8 Cauchy Integral Formula's

Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i} \int\limits_C \frac{f(z)}{(z-z_0)} \, \mathrm{d}z$$

$$f^n(z_0) = \frac{n!}{2\pi i} \oint\limits_C \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z$$

6.9 Liouville's Theorem

The only bounded entire functions are constants.

6.10 Cauchy's Inequality

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}$$

where M is a real number such that $|f(z)| \leq M$ for all points z on C, and C is the contour $|z - z_0| = r$.

7 Sequences and Series

$$\sum_{k=0}^{\infty} az^k = \lim_{k\to\infty} a + az + az^2 + az^3 + ... + az^k = \frac{a}{1-z}$$

7.1 Convergence/Divergence Tests

7.1.1 nth Term Test

If $lim_{n\to\infty} z_n \neq 0$, then the series $\lim_{k=1}^{\infty} z_k$ diverges.

7.1.2 Ratio Test

$$\lim_{n\to\infty} |\frac{z_{n+1}}{z_n}| = L$$

- If $L < 1$, series *converges absolutely*
- If $L > 1$ or $L = \infty$, the series *diverges*
- If $L = 1$, the test is inconclusive

7.1.3 Root Test

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$$

- If $L < 1$, series *converges absolutely*
- If $L > 1$ or $L = \infty$, the series *diverges*
- If $L = 1$, the test is inconclusive

7.2 Geometric Series

If $|z| < 1$, then

$$\sum_{k=0}^{\infty} az^k = \frac{a}{1-z}$$

7.3 Power Series

$$\sum_{k=0}^{\infty} a_k(z - z_0)^k$$

Represents an analytic function within its circle of convergence.

7.4 Taylor's Theorem

Let f be analytic within a domain D and let z_0 be a point in D. Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

7.5 Maclaurin Series

Taylor series centered at $z_0 = 0$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!}(z)^k$$

7.6 Laurent's Theorem

Let f be analytic within the annular domain D defined by $r < |z - z_0| < R$. Then, f has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$$

valid for $r < |z - z_0| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_c \frac{f(s)}{(s - z_0)^{k+1}} ds$$

where $k = 0, \pm 1, \pm 2, ...$, and C is a simply closed curve that lies entirely within D and has z_0 in its interior.

Note: Assuming f(z) analytic on domain D,

$$a_k = \frac{1}{2\pi i} \oint_c \frac{f(s)}{(s - z_0)^{k+1}} ds = \frac{1}{(k)!} \frac{d^k}{dz^k} f(z)|_{z=z_0}$$

7.7 Common Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ... + \frac{x^n}{(n)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

8 Poles, Zeros, Residues

8.1 Zeros

z_0 is a *zero* of a function f if $f(z_0) = 0$. An analytic function f has a zero of order n at $z = z_0$ if

$$f(z_0) = 0, f'(z_0) = 0, ..., f^{n-1}(z_0) = 0, \text{but } f^n(z_0) \neq 0$$

8.2 Singularities

Type of Singularities	Order	Laurent Series
Removable Singularity	$n = 0$	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$
Pole of Nth Order	$n = n$	$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + ... + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z - z_0) + ...$
Simple Pole	$n = 1$	$\frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z - z_0) + ...$
Essential Singularity	$n = \infty$	$... + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z - z_0) + ...$

8.3 Poles

If f and g are analytic at $z = z_0$ and f has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function $F(z) = \frac{g(z)}{f(z)}$ has a pole of order n at $z = z_0$.

8.4 Residue

$$\text{Res}(f(z), z_0) = a_{-1} = \frac{1}{2\pi i} \oint_C f(z)dz$$

Rearranging gives

$$2\pi i a_{-1} = 2\pi i \text{Res}(f(z), z_0) = \oint_C f(z)dz$$

8.5 Calculating Residue

8.5.1 Simple Pole

$$\text{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0)f(z)$$

8.5.2 Pole of Order N

$$\text{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

8.5.3 Non-Rational Functions

$$\text{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$$

8.6 Cauchy Residue Theorem

Let D be a simply connected domain and C be a simply closed curve inside D. Suppose $f(z)$ analytic on C and at region enclosed by C except at finitely many isolated singular points $z_1, z_2, ..., z_n$. Then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{N} \text{Res}(f(z), z_j)$$

9 Real Value Integrals

9.1 Trig Function Integrals

For integrals of the form:

$$\int_0^{2\pi} F(\cos(\theta)\sin(\theta)d\theta$$

Apply change of variables using $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$:

$$\oint_C F(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1}))\frac{dz}{iz}$$

Where C is $|z| = 1$ and $d\theta = \frac{dz}{iz}$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}(z + z^{-1})$$
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}(z - z^{-1})$$

9.2 Cauchy Principal Value

For integrals of the form:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{0} f(x)dx + \lim_{r\to\infty}\int_0^r f(x)dx$$

If both limits exist, the integral is *convergent*. Otherwise, integral is *divergent*.

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{r} f(x)dx$$

If the integral is *convergent*, then its P.V. (Principal Value) is equal to the value of the integral

9.2.1 Jordan Lemma

Suppose $f(z) = \frac{P(z)}{Q(z)}$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is m. C_r is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, then

If $m > n + 1$, then

$$\int_{C_r} f(z)dz = \int_{C_r} \frac{P(z)}{Q(z)}dz \to 0 \quad \text{as} \quad R \to \infty$$

If $m > n$ and $\alpha > 0$, then

$$\int_{C_r} f(z)e^{i\alpha z}dz = \int_{C_r} \frac{P(z)}{Q(z)}e^{i\alpha z}dz \to 0 \quad \text{as} \quad R \to \infty$$

9.3 Indented Contours

Suppose f has a simple pole $z = c$ on the real axis. If C_ϵ is the contour defined by $z = c + re^{i\theta}$ for $0 \leq \theta \leq \pi$, then

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)dz = \pi i \text{Res}(f(z), c)$$

If $\tilde{C}$ is the indented contour,

$$\oint_{\tilde{C}} f(z)dz = P.V. \int_{-\infty}^{\infty} f(x)dx + \lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)dz$$

$$P.V. \int_{-\infty}^{\infty} f(x)dx = \sum_{j=1}^{N} \text{Res}(f(z), z_j) - \pi i \text{Res}(f(z), c)$$

10 Absolute Value and Inequality

10.1

10.2 Triangle Inequality

$$||x| - |y|| \leq |x + y| \leq |x| + |y|$$