# 1 Coordinate Systems, Frames, Geometry

## 1.1 Points and Vectors

$$p^0 = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad v^0 = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$P = O_0 + P_x x_0 + P_y y_0 + P_z z_0$$

## 1.2 Rotation Matrices

$$R_0^1 = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 \end{bmatrix}$$

## 1.3 Properties of Rotation Matrices

$$R_0^1 = (R_1^0)^T = \begin{bmatrix} x_1^0 & y_1^0 & z_1^0 \end{bmatrix}$$

$$R_0^1 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$$v^i = R_j^i v^j \qquad v^0 = R_1^0 v_1$$

## 1.4 Orthogonality

$$R_1^0 = (R_0^1)^T = (R_1^0)^{-1} \qquad R^T = R^{-1}$$

$$\det(RR^T) = \det(I) = 1 \implies \det(R)^2 = 1$$

## 1.5 Elementary Rotations

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 1.6 Compositions of Rotations

### 1.6.1 Case 1: Sequential Transformations

Let $R$ be a coordinate transformation in F1

$$R_2^0 = R_1^0 \cdot R_2^1 = R_1^0 \cdot R$$

### 1.6.2 Case 2: Global Transformations

Let $R$ be a coordinate transformation in F0

$$R_2^0 = R_1^0 \cdot R_2^1 = R \cdot R_1^0$$

# 2 Euler Angles

$$R_0^1 = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

$$R_0^1 = R_{z,\phi} \cdot R_{y,\theta} \cdot R_{z,\psi} \qquad r = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

## 2.1 Case $s_\theta > 0$

$$\theta = \operatorname{atan2}\left(r_{33}, \sqrt{1 - r_{33}^2}\right)$$

$$\phi = \operatorname{atan2}(r_{13}, r_{23}) \qquad \psi = \operatorname{atan2}(-r_{31}, r_{32})$$

## 2.2 Case $s_\theta < 0$

$$\theta = \operatorname{atan2}\left(r_{33}, -\sqrt{1 - r_{33}^2}\right)$$

$$\phi = \operatorname{atan2}(-r_{13}, -r_{23}) \qquad \psi = \operatorname{atan2}(r_{31}, -r_{32})$$

# 3 Homogenous Transformation Matrix

$$H := \begin{bmatrix} R_{11} & R_{12} & R_{13} & d_x \\ R_{21} & R_{22} & R_{23} & d_y \\ R_{31} & R_{32} & R_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & d \\ 0_3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} p^0 \\ 1 \end{bmatrix} = H_1^0 \begin{bmatrix} p^1 \\ 1 \end{bmatrix} \qquad H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

# 4 Forward Kinematics

## 4.1 Exceptions to DH Convention

1. $l_{i-1}, l_i$ parallel
   (a) Infinite common normals: pick any $O_i \in l_i$
2. $l_{i-1}, l_i$ have a unique point of intersection
   (a) Set $O_i = l_i \cap l_{i-1}$, choose $x_i \perp (z_{i-1}), x_i \perp (z_i)$
   (b) $x_i = \pm(z_{i-1} \times z_i)$
3. $l_i = l_{i-1}$
   (a) Choose $O_{i-1}$ to be any point on $l_i$, $x_i \perp z_i$

## 4.2 DH Parameters

- $d_i$: displacement between $O_{i-1}, O_i$ along $z_{i-1}$
- $a_i$: length of common normal between $l_{i-1}$ and $l_i$ (along $x_i$ axis)
- $\theta_i$: angle from $x_{i-1}$ to $x_i$ measured as RH rotation about $z_{i-1}$
- $\alpha_i$: angle from $z_{i-1}$ to $z_i$ measured as RH rotation about $x_i$

## 4.3 Consecutive Joint Homogeneous Transforms

$$H_i^{i-1} = \begin{bmatrix} R_i^{i-1} & O_i^{i-1} \\ 0_3 & 1 \end{bmatrix}$$

$$H_i^{i-1} = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# 5 Inverse Kinematics

$$H_d = \begin{bmatrix} R_d & O_d \\ 0 & 1 \end{bmatrix}$$

Find $q_1, \ldots, q_n$ s.t. $H_n^0(q_1, \ldots, q_n) = H_d$

1. Case 1: $n > 6$
   (a) infinite solutions (redundant robot)
2. Case 2: $n = 6$
   (a) Finite amount of solutions
3. Case 3: $n < 6$
   (a) No solutions

## 5.1 Kinematic Decoupling

$$O_6^0 = O_C^0 + d_6 \cdot z_6 \qquad O_C^0 = O_6^0 - d_6 \cdot R_6^0 z_0$$

Find $q_1, q_2, q_3$ s.t. $O_C^0(q_1, q_2, q_3) = O_6^0 - d_6 \cdot R_6^0 z_0$. Then compute $R_3^0(q_1, q_2, q_3)$. Then, notice $R_6^0 = R_0^3 \cdot R_6^3$ and calculate:

$$R_6^3 = [R_3^0]^T R_d$$

# 6 Velocity Kinematics

$$p^0 = R_1^0 p^1 + O_1^0 \qquad \dot{p}^0 = \dot{R}_1^0 p^1 + \dot{O}_1^0$$

## 6.1 Skew-Symmetric Matrices

Given $w = (\begin{bmatrix} w_x & w_y & w_z \end{bmatrix})^T$,

$$S(w) = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$$

### 6.1.1 Properties of Skew-Symmetric Matrices

$$S(\alpha \cdot a + \beta \cdot b) = \alpha S(a) + \beta S(b)$$

$$S(a)p = a \times p \qquad RS(a)R^T = S(Ra)$$

$$S^T + S = 0 \qquad S^T = -S$$

## 6.2 Angular Velocity

$$\dot{R}(t) = S(\omega(t))R(t) \qquad \dot{R}_1^0 (R_1^0)^T = S(\omega_1^0)$$

### 6.2.1 Special Case: Fixed Axis

$$\dot{p}^0 = \omega_1^0 \times (R_1^0 p^1) = S(\omega_1^0) R_1^0 p^1 \qquad p^0 = R_1^0 p^1$$

### 6.2.2 Instantaneous Axis of Rotation

$$l = \{q^0 \in \mathbb{R} : q^0 = O_1^0 + \lambda w_1^0, \lambda \in \mathbb{R}\}$$

$$R_1^0 p^1 = \lambda \omega_1^0$$

## 6.3 Composition of Angular Velocities

$$\dot{R}_2^0 = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1 = S(\omega_1^0 + R_1^0 \omega_2^1) R_2^0$$

$$\omega_2^0 = \omega_1^0 + R_1^0 \omega_2^1 \qquad \omega_n^0 = \omega_1^0 + R_1^0 \omega_2^1 + \ldots + R_{n-1}^0 \omega_n^{n-1}$$

# 7 Robot Jacobian

Suppose $p^0(t) = F(q(t))$. Then $\dot{p}^0(t) = \frac{\partial F}{\partial q}(q(t)) \cdot \dot{q}(t)$

$$J(q) = \frac{\partial F}{\partial q}(q(t)) \qquad J(q) \cdot \dot{q} = \begin{bmatrix} \dot{O}_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v(q) \\ J_\omega(q) \end{bmatrix} \cdot \dot{q}$$

## 7.1 Linear Velocity Jacobian

$$J_v^i(q) = \begin{cases} z_{i-1}^0 & \text{joint i is P} \\ z_{i-1}^0 \times (O_n^0 - O_{i-1}^0) & \text{joint i is R} \end{cases}$$

$$J_v = \begin{bmatrix} J_v^1 & J_v^2 & \ldots & J_v^n \end{bmatrix}$$

## 7.2 Angular Velocity Jacobian

$$J_\omega^i(q) = \begin{cases} 0 & \text{joint i is P} \\ z_{i-1}^0 & \text{joint i is R} \end{cases}$$

$$J_\omega = \begin{bmatrix} J_\omega^1 & J_\omega^2 & \ldots & J_\omega^n \end{bmatrix}$$

# 8 Inverse Velocity Kinematics

Given $\xi^0 = \begin{bmatrix} \dot{O}_n^0 \\ \omega_n^0 \end{bmatrix}$, find $\dot{q}$

1. Case 1: $n > 6$
   (a) Solvable iff $\operatorname{rank}(J(q)) = 6$
   (b) Infinite solutions
2. Case 2: $n = 6$
   (a) Solvable iff $J(q)$ is invertible and has unique solution
   (b) $\dot{q} = J(q)^{-1} \xi^0$ $(\operatorname{rank}(J(q)) = 6)$
3. Case 3: $n < 6$
   (a) No solutions

### 8.0.1 Right Pseudoinverse Solution

$$\dot{q} = J^+(q)\xi^0 \qquad J^+(q) = J(q)^T (J(q)J(q))^{-1}$$

$$\dot{q} = J^+(q)\xi^0 + (I_6 - J^+(q)J(q))b \quad \forall b \in \mathbb{R}^n$$

# 9 Force/Torque Relationship

$$\tau = J(q)^T F^0$$

# 10 Kinematic Singularities

For a matrix $J \in \mathbb{R}^{6xn}$, $\operatorname{rank}(J(q)) \leq \min(6, n)$. A joint vector $q$ is a kinematic singularity if

$$\operatorname{rank}(J(q)) < \max \operatorname{rank}(J(q))$$

## 10.1 n=6 case

$$\text{Singular if } \det(J(q)) = 0$$

$$J \in \mathbb{R}^{6xn} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{bmatrix}$$

$$\det(J) = \det(J_{11})\det(J_{22}) = 0$$

# 11 Robot Modelling

## 11.1 Holonomic Constraints

A holonomic constraint for a sys of $N$ particles and $l$ constraints is a relation $g(r_i, \ldots, r_N) = 0$

$$g : \mathbb{R}^3 \times \mathbb{R}^3 \times \ldots \times \mathbb{R}^3 \to \mathbb{R}^l$$

s.t. $g$ differentiable, $\frac{\partial g}{\partial r}$ full row rank $l$ at each r.

$$L = \{r \in \mathbb{R}^{3N} : g(r) = 0\}$$

## 11.2 Constraint Reaction Forces

$$f_c \cdot \delta_r = (\lambda r) \cdot dr = \lambda(r \cdot dr) = 0$$

## 11.3 Generalized Coordinates

$$r = r(q_1, \ldots, q_n)$$

$(q_1, \ldots, q_n)$ are the generalized coordinates

## 11.4 Degrees of Freedom

$$\#\,\text{DoF} = n := 3 \cdot N - l$$

## 11.5 Parametric Representation

$$L = \{r(q) : q \in \mathbb{R}\}$$

## 11.6 Virtual Displacement

$$\delta r \in \mathbb{R}^{3N} \quad , \quad \delta r = \begin{bmatrix} \delta r^1 \\ \vdots \\ \delta r^N \end{bmatrix}$$

$$\delta r := \delta r \perp r \quad \{r \in \mathbb{R}^2 : ||r|| = l\}$$

$$r \cdot dr = 0 \qquad \frac{\partial g}{\partial r}\delta r = 0 \qquad \delta r = \frac{\partial r}{\partial q} dq$$

## 11.7 Lagrange D'Alembert Principle

$$(M\ddot{r} - f_L) \cdot \delta_r - f_c \cdot \delta r = 0$$

## 11.8 Generalized Force

$$\psi := \left[\frac{\partial r}{\partial q}\right]^T f_L$$

$$f_\psi = -\nabla_r U + f_a \qquad \psi = -\nabla_q P + \tau$$

Where $f_a$ is the app. force and $\tau$ is the generalized app. force

# 12 Euler Lagrange Equation

$$\frac{d}{dt}\nabla_{\dot{q}} \mathcal{L} - \nabla_q \mathcal{L} = \tau$$

$$\tau := \left(\frac{\partial r}{\partial q}\right)^T f_a = \sum_i \left(\frac{\partial r^i}{\partial q}\right)^T f_a^i$$

## 12.1 Lagragian Equation

$$\mathcal{L}\{q, \dot{q}\} := K(q, \dot{q}) - P(q) = K - P$$

## 12.2 Point Masses

### 12.2.1 Kinetic Energy

$$K = \sum_{i=1}^N K_i = \sum_{i=1}^N \frac{1}{2} m_i ||\dot{r}_i||^2$$

### 12.2.2 Potential Energy

$$P_i = m_i \cdot g \cdot h_i$$

## 12.3 Distributed Mass Systems

### 12.3.1 Center of Mass

$$r_c^0 := \frac{\sum m_i r_i^0}{\sum m_i}$$

### 12.3.2 Mass Moment of Inertia

$$I := \sum_i m_i S(d_i^0)^T S(d_i^0) = -\sum_i m_i S(d_i^0)^2$$

$$I = \begin{bmatrix} \sum m_i (y_i + z_i)^2 & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ \sum m_i x_i y_i & m_i (x_i + z_i)^2 & -\sum m_i y_i z_i \\ \sum m_i x_i z_i & \sum m_i y_i z_i & \sum m_i (x_i + y_i)^2 \end{bmatrix}$$

### 12.3.3 Kinetic Energy

$$\dot{r}_i^0 = \dot{r}_c^0 - d_i^0 \times \omega_1^0$$

$$K_i = \frac{1}{2} m_i ||\dot{r}_i||^2 + \frac{1}{2}(\omega_1^0) \cdot I \cdot \omega_1^0$$

# 13 Robot Models

## 13.1 Basic (Lagrangian) Model

$$J_\omega^i = \begin{bmatrix} \rho_1 z_0^0 & \ldots & \rho_i z_i^0 & | & O_{3 \cdot (n-1)} \end{bmatrix}$$

$$J_v^i = \begin{bmatrix} z_0^0 \times O_i^0 & \ldots & z_{i-1}^0 \times (O_i^0 - O_{i-1}^0) & | & O_{3 \cdot n} \end{bmatrix}$$

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^T \left[\sum_i \left(M_i J_v^i(q)^T J_v^i(q) + J_\omega^i(q)^T I_i J_\omega^i(q)\right)\right]$$

$$P(q) = \sum_{i=1}^n -M_i (g^0)^T r_{c_i}^0$$

## 13.2 Christoffel Coefficients

$$C_{ijk}(q) = \frac{\partial d_{ik}}{\partial q_j} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k} = \frac{1}{2}\left[\frac{\partial_{kj}}{\partial q_i} + \frac{\partial_{ki}}{\partial q_j} - \frac{\partial_{ij}}{\partial q_k}\right]$$

$$[C(q, \dot{q})]_{kj} = \sum_{i=1}^N C_{ijk}(q)\dot{q}_i$$

## 13.3 Basic (Lagragian) Model EOMs

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla_q P = \tau$$

## 13.4 Control (Enhanced) Model EOMs

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B(q)\dot{q} + \nabla_q P = u$$

$$M(q) = D(q) + \begin{bmatrix} r_1^2 J_{m_1} & & \\ & \ddots & \\ & & r_n^2 J_{m_n} \end{bmatrix}$$

$$u_i = r_i \frac{K_{m_i}}{R_i} v_i$$

# 14 Stability of NL Systems

## 14.1 Positive, Negative, Definite, Semidefinite

A differentiable function $V : \mathbb{R}^n \to \mathbb{R}^n$ is p.d. at $\bar{x}$ if $V(x) > 0 \forall x \neq \bar{x}$ and $V(\bar{x}) = 0$.
A differentiable function $V$ is n.d. at $\bar{x}$ if $-V(x)$ is p.d. and $V(\bar{x}) = 0$.
A differentiable function $V$ is ps.d. at $\bar{x}$ if $V(x) \geq 0$ and $V(\bar{x}) = 0$.
A differentiable function $V$ is ns.d. at $\bar{x}$ if $V(x) \leq 0$ and $V(\bar{x}) = 0$.

## 14.2 Positive Definite Theorem

P is p.d. if all principal leading minors are +ve

$$M_1 = P_{11} \quad M_2 = \det\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad M_n = \det P$$

## 14.3 Lyapunov Theorem

Suppose $\bar{x} \in \mathbb{R}^n$ is an equilibrium of $\dot{x} = f(x)$, and $\exists V : \mathbb{R}^N \to \mathbb{R}$ which is p.d. at $\bar{x}$ s.t. $\dot{V} = \frac{\partial V}{\partial x} f(x)$ is ns.d. Then $\bar{x}$ is a stable equilibrium.
If $\dot{V}$ is n.d. at $\bar{x}$, then it is asy. stable.

### 14.3.1 Lyapunov Functions

Mass-Spring Damper System:

$$\dot{x_1} = x_2 \quad \dot{x_2} = -\frac{k}{m}x_1 - \frac{b}{m}x_2$$

$$V(x) = \frac{1}{2}(x_1^2 + x_1 x_2 + x_2^2)$$

General

$$V(q, \dot{q}) = K + P = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}\tilde{q}^T K_p \tilde{q}$$

Energy of Tracking Error (Used for passivity controller)

$$V = \frac{1}{2}r^T M(q) r + \tilde{q}^T P \tilde{q}$$

## 14.4 Karsovski-LaSalle Invariance Principle (KL)

Let $\bar{x}$ be an equilibrium of $\dot{x} = f(x)$, and suppose $\exists V : \mathbb{R}^N \to \mathbb{R}$ which is p.d. at $\bar{x}$ and s.t. $\dot{V} = \frac{\partial V}{\partial x} f(x)$ is ns.d. Then, $\bar{x}$ is stable and $\dot{V}(x(t)) \to 0$ as $t \to \infty$. Moreover, if $\dot{V}(x(t)) \equiv 0 \forall t$ implies $x(t) \equiv \bar{x}$, then $\bar{x}$ is asy. stable.

# 15 Robot Control

$$\tilde{q} = q^r - q \qquad \dot{\tilde{q}} = \dot{q}^r - \dot{q}$$

## 15.1 Decentralized Model

$$J_m \ddot{\theta}_m + B_m \dot{\theta}_m = \tau_m - \tau_l$$

$$\tau_m = K_m \cdot i_a \quad \text{Assume } \frac{L}{R} << \frac{J_m}{B_m}$$

$$L\frac{di_a}{dt} + Ri_a = v - K_b \dot{\theta}_m$$

$$i_a \approx \frac{V}{R} - \frac{K_b}{R}\dot{\theta}_m$$

$$J := J_m \; B := B_m + \frac{K_m K_b}{2} \; u := \frac{K_m}{R} V$$

$$G(s) = \frac{1}{Js^2 + Bs} \qquad C(s) = (K_p + K_d \cdot s)$$

## 15.2 Feedback Linearization (Computed Torque)

Goal: Find $a_i (= \ddot{q}_i)$ s.t. $q_i(t) \to q_i^r(t)$

$$\tilde{q}_i(t) = q_i^r(t) - q_i(t)$$

$$M(q)a + C(q, \dot{q}) + B(q)\dot{q} + \nabla_q P = u$$

$$a = \ddot{q}^r(t) + K_p \tilde{q} + K_d \dot{\tilde{q}}$$

$$K_p = \begin{bmatrix} K_{p_1} & & \\ & \ddots & \\ & & K_{p_n} \end{bmatrix} \quad K_d = \begin{bmatrix} K_{d_1} & & \\ & \ddots & \\ & & K_{d_n} \end{bmatrix}$$

## 15.3 PD Control with Gravity Compensation

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B(q)\dot{q} + \nabla_q P = u$$

$$u = K_p \tilde{q} + K_d \dot{\tilde{q}} + \nabla_q P$$

## 15.4 Passivity Based Controller (Slotine-Li)

$$r(t) = \dot{\tilde{q}}(t) + \Lambda \tilde{q}(t)$$

$$r(t) \to 0 \text{ as } t \to \infty \quad r(t) \equiv 0 \iff \dot{\tilde{q}} = -\Lambda \tilde{q}$$

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B(q)\dot{q} + \nabla_q P = u$$

$$u = M(q)(\ddot{q}^r + \Lambda \dot{\tilde{q}}) + C(q, \dot{q})(\dot{q}^r + \Lambda \tilde{q}) + B(q)\dot{q} + \nabla_q P + K(\dot{\tilde{q}} + \Lambda \tilde{q})$$

$$M(q)\dot{r} + C(q, \dot{q})r + Kr = 0 \quad r := \dot{\tilde{q}} + \Lambda \tilde{q}$$

$$K = K^T \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \lambda_i > 0$$

# 16 Trigonometric Identities

$$\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$