ECE311 Course Notes

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October 2021

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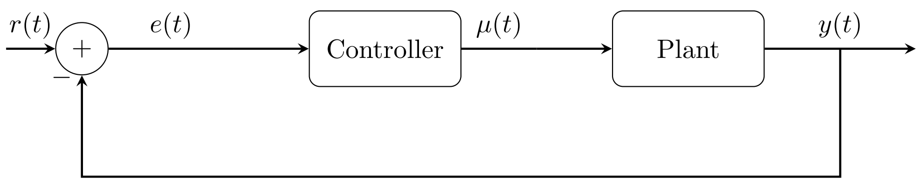

Where

- $\mu(t)$ is the control input (decision variable)
- $y(t)$ is the output variable (measured with sensors and also the target of our control)
- $r(t)$ is the reference signal. We want $y(t) \to r(t)$ as $t \to \infty$
- $e(t)$ is the tracking error. We want $e(t) \to 0$ as $t \to \infty$

1 Signals

1.1 Time Constant

$$e^{-A \cdot t} \leftrightarrow e^{-t/\tau} \to \tau = \frac{1}{A}$$

2 Control System Models

2.1 Non-Linear Time Invariant State Space (NN)

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

$$\dot{x} = f(x, u) = f(x_1, ..., x_n, u_1, ..., u_n)$$
$$y = h(x, u) = h(x_1, ..., x_n, u_1, ..., u_n)$$

2.2 LTI State Space Models

$$\dot{x_i} = a_{i1}x_1 + ... + a_{in}x_n + b_{i1}u_1 + ... + b_{im}u_m$$
$$\dot{y_j} = c_{j1}x_1 + ... + a_{jn}x_n + d_{j1}u_1 + ... + d_{im}u_m$$
$$\dot{x} = Ax + Bu \quad y = Cx + Du \quad x = [x_1 \cdots x_n]^T$$

2.3 LTI Input/Output Models (I/O) Models

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y =$$
$$b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u$$

for $m \leq n$

3 Equilibrium

For a NN system, a state $\bar{x} \in \mathbb{R}$ is an **equilibrium** if

$$f(\bar{x}, \bar{u}) = [0, \cdots, 0]^T$$

4 Linearization

$$\bar{x} = [\bar{x}_1, \cdots, \bar{x_n}]^T, \quad \bar{u}$$
$$\tilde{x} \coloneqq x - \bar{x} \qquad \tilde{u} \coloneqq u - \bar{u} \qquad \tilde{y} \coloneqq y - h(\bar{x}, \bar{u})$$

Then, the **linearization** of $\bar{x}$ is given by

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} \qquad \tilde{y} = C\tilde{x} + D\tilde{u}$$
$$A = \left[\left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})}\right] \qquad B = \left[\left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})}\right]$$
$$C = \left[\left.\frac{\partial h}{\partial x}\right|_{(\bar{x},\bar{u})}\right] \qquad D = \left[\left.\frac{\partial h}{\partial u}\right|_{(\bar{x},\bar{u})}\right]$$

5 Matrix Inverses

$$A^{-1} = \frac{\text{adj}(A)}{\det A} = \frac{(C)^T}{\det A}, \quad C_{ij} = (-1)^{i+j}M_{ij}$$
$$\text{adj}(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\operatorname{adj}(A) = \begin{pmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix} \tag{1}$$

6 Final Value Theorem

If $\lim_{t\to\infty} f(t)$ exists, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

6.1 FVT Existence Condition

A signal $f(t)$ is bounded iff $F(s)$ has poles with real part ≤ 0 and non-repeated poles with real part $= 0$

7 Initial Value Theorem

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

8 Laplace Transform (LT)

Let $f(t)$ be a function $f : \mathbb{R} \to \mathbb{R}$. Then

$$\mathscr{L}\{f(t)\} = F(s) := \int_0^{+\infty} f(t)e^{-st}dt$$

Where $F : \mathbb{C} \to \mathbb{C}$. The LT exists if

1. $f(t)$ is Piecewise Continuous (PWC)
2. $\exists M \geq 0, a \in \mathbb{R}$ s.t. $|f(t)| \leq Me^{at}\ \forall t \geq 0$

8.1 Basic Laplace Table

$\mathscr{L}\{1(t)\} = \frac{1}{s}$	$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$
$\mathscr{L}\{\frac{t^k}{k!}e^{at}\} = \frac{1}{(s-a)^{k+1}}$	$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$
$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$	$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$
$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$	$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$

8.2 First Translation Theorem

$$\mathscr{L}\{e^{at}f(t)\}(s) = \mathscr{L}\{f(t)\}(s-a) = F(s-a)$$

8.3 Second Translation Theorem

$$\mathscr{L}\{f(t-a)\mathrm{u}(t-a)\} = e^{-as}F(s)$$

where u is the unit step function and $a > 0$.

8.4 Transforms of Derivatives

If $f, f', ...f^{(n-1)}$ are cts on $[0, \infty)$ and are of expon. order, and if $f^{(n)}(t)$ is piecewise cts on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

8.5 Derivatives of Transforms

If $\mathscr{L}\{f(t)\} = F(s)$ and $n = 1, 2, 3, ...$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

8.6 Transform of Integrals

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \qquad \int_0^t f(\tau)d\tau = \mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}$$

9 Inverse Laplace Transform

9.1 Basic Inverse Laplace Transforms

$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$	$\mathscr{L}^{-1}\{1\} = \delta(t)$
$\mathscr{L}\left\{\frac{1}{s-a}\right\} = e^{at}$	$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$
$\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$	$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$
$\mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh(kt)$	$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

9.2 Inverse Laplace Transform Formula

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(z)e^{zt}dt = \sum_{k=1}^{n} \mathrm{Res}(e^{st}F(s), s_k)$$

9.3 Residue Calculation

In general, the residue of a function $F(s)$ at a pole can be calculated with

$$\text{Res}(F(s), s_k) = \frac{1}{(n-1)!} \lim_{s \to s_k} \frac{d^{n-1}}{ds^{n-1}} (s - s_0)^n F(s)$$

Where $n \geq 1$ is the order of the function $F(s)$.

10 Model Conversions

10.1 Input/Output to Transfer Function

For an IO model of the form

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + \cdots + b_0 u$$

with $y(0) = \dot{y}(0) = \ddot{y}(0) = \cdots = 0$, the equivalent Transfer Function model is given by

$$G(s) = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

10.2 Transfer Function to Input/Output

For a Transfer Function model of the form,

$$Y(s) = G(s)U(s)$$

the equivalent Input/Output model is given by

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\{G(s)U(s)\}$$
$$y(t) = g(t) * u(t) = \int_0^t g(t - \tau) u(\tau) d\tau$$

10.3 State Space to Transfer Function

For a State Space model of the form,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

the equivalent Transfer Function model is given by

$$Y(s) = [C(sI - A)^{-1} B + D] U(s)$$
$$G(S) = C(sI - A)^{-1} B + D$$

10.3.1 Notes

The values of $S \in \mathbb{C}$ for which $sI - A$ is not invertible are poles of $G(s)$

10.4 Transfer Function to State Space

$$V(s) = \frac{1}{s^n + \cdots + a_0} U(s)\ ,\ Y(s) = (b_m s^m + \cdots + b_0) V(s)$$

$$f(x,u) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_0\ b_1\ \cdots\ b_m\ 0\ \cdots\ 0]x$$

11 Poles

$$\text{1st Order: } (s + p_1) \quad \text{2nd Order: } [(s+\sigma)^2 + \omega_d^2]$$

11.1 Pole Poly Representations

$$\frac{as + b}{[(s+\sigma)^2 + \omega_d^2]} \leftrightarrow \frac{as+b}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\sigma = \zeta\omega_n \quad \omega_d = \sqrt{\omega_n^2 - \zeta^2\omega_n^2} = \sqrt{1-\zeta^2}$$

$$\zeta = \frac{\sigma}{\sqrt{\sigma^2 + \omega_d^2}} \quad \omega_n = \sqrt{\sigma^2 + \omega_d^2}$$

12 Transient Performance

12.1 2nd Order Systems

12.1.1 Settling Time

$$T_s \approx -\frac{\ln(2 \cdot 10^{-2}\sqrt{1-\zeta^2})}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma}$$

12.1.2 Percent Overshoot and Peak Time

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1-\zeta^2}}, \zeta = \frac{-\ln(\%\text{OS})}{\sqrt{\pi^2 + \ln^2(\%\text{OS})}}$$

$$\%\text{OS} = y(T_p) - 1 = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

12.1.3 Rise Time

$$T_r\omega_n \propto \zeta \qquad \longrightarrow \qquad T_r\omega_n \approx \frac{1.8}{\omega_n}$$

12.1.4 Effect of Additional Pole/Zeroes

Additional Pole/Zeroes in LHP have little effect, as long as $Re\{P\} << \sigma \to Re\{P\} \leq 10 \cdot \sigma$. Zeroes in RHP (**Nonminimum Phase**) and change sign of $y(\infty)$.

12.2 Higher Order (Dominant Pole)

12.2.1 Phase Margin

$$\text{PM} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}\right)$$
$$\text{PM} \approx 100\zeta \quad \text{for } 0 \leq \zeta \leq 0.6$$

12.2.2 Bandwidth

$$\omega_{BW} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2(1-\zeta)^2}}$$
$$T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\zeta\omega_{BW}} \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2(1-\zeta)^2}}$$

12.2.3 Crossover Frequency

$$\omega_c = \omega_{BW} \frac{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}{\sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2(1-\zeta^2)}}} \approx 0.635\omega_{BW}$$
$$\omega_c \approx 0.5 \cdot \omega_{BW} \qquad \omega_c \leq \omega_{BW} \leq 2\omega_c$$

13 Stability

13.1 Internal Stability

A system is **Internally Stable** if $\forall x_o \in \mathbb{R}$ the solution of $\dot{x} = Ax$ with I.C $x(0) = x_0$ is **bounded**.

13.2 Asymptotic Stability (AS)

A system is **asymptotically stable** if $\forall x_o \in \mathbb{R}^n$ with I.C. $x(0) = x_0$, $x(t) \to 0$ as $t \to \infty$.

$$\dot{x} = Ax \quad X(s) = \frac{\text{Adj}(sI - A)x_0}{\det(sI - A)}$$

AS if all poles (all eigenvalues) of $X(s)$ in OLHP.

13.3 Input/Output Stability (BIBO)

A system is **BIBO Stable** if for any bounded input $x(t)$, the output $y(t)$ is also bounded.

$$Y(s) = G(s)U(s) \quad G(s) = \frac{C\text{Adj}(sI - A)B + D}{\det(sI - A)}$$

BIBO Stable if all poles of $G(s)$ in OLHP.

13.4 Routh Array

\# of sign variations = # of roots with real part < 0

$$b_1 = \frac{-1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{bmatrix} \quad b_2 = \frac{-1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{bmatrix}$$

$$c_1 = \frac{-1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{bmatrix} \quad c_2 = \frac{-1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{bmatrix}$$

14 Basic (Standard) Control Problem

$$E(s) = \frac{1}{1+CG}R(s) + \frac{-G}{1+CG}D(s) = E_R \cdot R + E_D \cdot D$$

$$U(s) = \frac{C}{1+CG}R(s) + \frac{1}{1+CG}D(s)$$

Closed Loop System BIBO Stable if G4 BIBO Stable.

14.1 G4 Stability

1. No illegal pole/zero cancellations in CG
2. Zeroes of $1 + CG$ in OLHP

14.2 Type

A TF has type l if it has exactly l poles at 0. Suppose $R(s)$ has type K. If CG has type $K-1$, then $e(\infty)$ is nonzero, finite. If CG has type $K-2$, then $e(\infty)$ is unbounded.

15 Internal Model Principle (IMP)

$R(s), D(s)$ rational, strictly proper. Then $e(t) \to 0$ iff 1. G4 BIBO Stable 2. Poles of R are also poles of CG (CG type K_R) 3. Poles of D are also poles of C (C type K_D)

16 Controllers

16.1 PD Controllers

Not physically implementable (unless $\dot{y}(t)$ sensor).

$$C(s) = K(T_d \cdot s + 1) \quad \leftrightarrow \quad u(t) = Ke(t) + KT_d\dot{e}(t)$$

Use to increase the PM (by a max of $\pi/2$ by placing $\frac{1}{T_d}$ before the ω_c

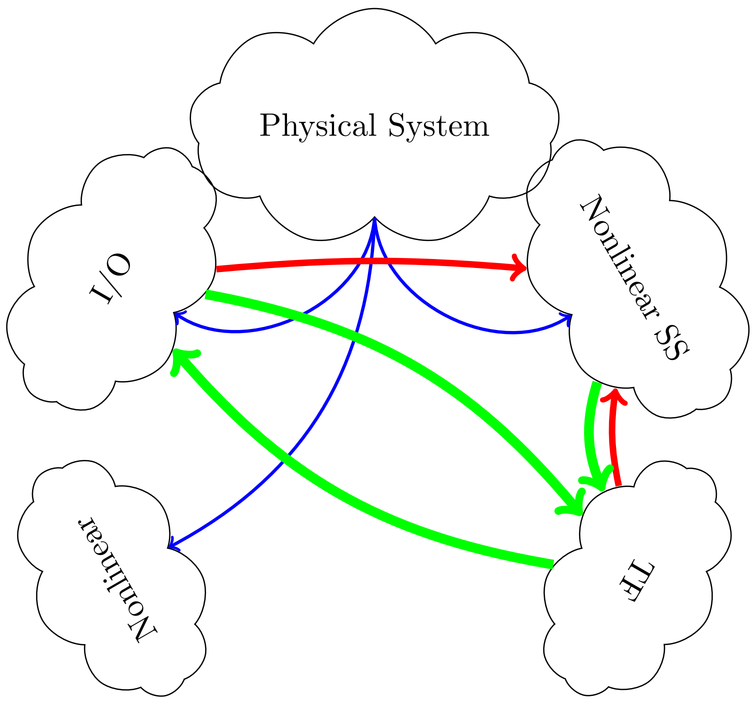

Figure 1: Blue arrows represent conversions from a physical model to a mathematical model. Green arrows represent unique conversions between mathematical models of systems. Red arrows represent non-unique conversions between mathematical models.