ECE221 Course Notes

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Chapter 1

Introduction and Calculus

1.1 Basic Integrals

$$\int \frac{1}{(x^2+y^2+z^2)^{3/2}}dx = \frac{x}{(y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$$\int_{\infty}^{\infty} \frac{1}{(x^2+y^2+z^2)^{3/2}}dx = \frac{2}{y^2+z^2}$$

1.2 Gradient

$$\frac{q}{4\epsilon_0\pi} \times \left[\frac{y-y_1'}{[(x-x_1')^2+(y-y_1')^2]^{3/2}} - \frac{y+y_1'}{[(x-x_1')^2+(y+y_1')^2]^{3/2}} - \frac{y-y_2'}{[(x-x_2')^2+(y-y_2')^2]^{3/2}} + \frac{y+y_2'}{[(x-x_2')^2+(y+y_2')^2]^{3/2}}\right]\hat{y} \tag{1.1}$$

1.2.1 Cartesian

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

1.2.2 Cylindrical

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{\partial}{\partial z}\hat{z}$$

1.2.3 Spherical

$$\nabla = \frac{\partial}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{R\sin(\theta)}\frac{\partial}{\partial \phi}\hat{\phi}$$

1.2.4 Gradient Properties

$$\nabla(U+V) = \nabla U + \nabla V$$
$$\nabla(UV) = U\nabla V + V\nabla U$$
$$\nabla V^n = nV^{n-1}\nabla V \quad \text{for any n}$$

1.2.5 Divergence Properties

Cylindrical

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Cylindrical

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Spherical

$$\nabla \cdot A = \frac{1}{R^2}\frac{\partial}{\partial r}(R^2 A_R) + \frac{1}{R\sin(\theta)}\frac{\partial}{\partial \theta}(A_\theta \sin(\theta)) + \frac{1}{R\sin(\theta)}\frac{\partial A_\phi}{\partial \phi}$$

1.2.6 Curl Properties

Cartesian

$$\nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Cylindrical

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{r} & \hat{\phi}r & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

Spherical

$$\nabla \times A = \frac{1}{R^2\sin(\theta)}\begin{vmatrix} \hat{R} & \hat{\theta}R & \hat{\phi}R\sin(\theta) \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin(\theta))A_\phi \end{vmatrix}$$

1.2.7 Laplacian Properties

Cartesian

$$\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$$

Cylindrical

$$\nabla^2 A = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \phi^2} + \frac{\partial^2 A}{\partial z^2}$$

Spherical

$$\nabla^2 A = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial A}{\partial R}\right) + \frac{1}{R^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial A}{\partial \theta}\right) + \frac{1}{R^2\sin^2(\theta)}\frac{\partial^2 A}{\partial \phi^2}$$

1.3 Surface Integrals

Let f be a continuous scalar-valued function on a smooth surface S given parametrically by $r(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where u and v vary over $R = \{(u,v) : a \le u \le b, c \le v \le d\}$. Assume also that the tangent vectors

$$t_u = \frac{\partial x}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right\rangle$$
$$t_v = \frac{\partial x}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right\rangle$$

are continuous on R and the normal vector $t_u \times t_v$ is nonzero on R. Then the surface integral of f over S is

$$\iint_S f(x,y,z)dS = \iint_R f(x(u,v), y(u,v), z(u,v)|t_u \times t_v|dA$$

1.3.1 Surface Area

$$\text{Surface Area} = \iint_S 1dS = \iint_R 1|t_u \times t_v|dA$$

1.4 Curl and Circulation

$$\text{Circ} = \oint_C F \cdot Tds$$

$$\text{Curl} = \nabla \times F = \lim_{A\to 0}\frac{\oint_C F \cdot Tds}{A}$$

where A is the area enclosed by contour C

$$\text{Curl} = \nabla \times F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \times \left\langle f(x,y,z), g(x,y,z), h(x,y,z)\right\rangle$$

1.5 Divergence and Flux

$$\text{Flux} = \oint_C F \cdot nds$$

$$\text{Div} = \nabla \cdot F = \lim_{A\to 0}\frac{\oint_C F \cdot nds}{A}$$

where A is the area enclosed by contour C

$$\text{Div} = \nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle f(x,y,z), g(x,y,z), h(x,y,z) \right\rangle$$

1.6 Vector Identities

1.6.1 Dot Product

$$A \cdot B = \langle A_1, A_2, A_3 \rangle \cdot \langle B_1, B_2, B_3 \rangle = A_1 B_1 + A_2 B_2 + A_3 B_3$$

1.6.2 Cross Product

$$A \times B = \langle A_1, A_2, A_3 \rangle \times \langle B_1, B_2, B_3 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

1.6.3 Scalar Triple Product

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

1.6.4 Divergence/Curl Linearity

$$\nabla \cdot (A + B) = \nabla \cdot A + \nabla \cdot B$$

$$\nabla \cdot (A + B) = \nabla \times A + \nabla \times B$$

1.6.5 Second Derivatives

Source Free Field

$$\nabla \cdot (\nabla \times A) = 0$$

Rotation Free Field

$$\nabla \times (\nabla \Psi) = 0$$

Scalar Laplacian

$$\nabla \cdot (\nabla \Psi) = \nabla^2 \Psi$$

Vector Laplacian

$$\nabla(\nabla \cdot A) - \nabla \times (\nabla \times A) = \nabla^2 A$$

1.7 Stokes Theorem

$$\text{circ}(F) = \oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n dS$$

1.8 Divergence Theorem

$$\text{flux}(F) = \iint_S F \cdot nds = \iiint_D (\nabla \cdot F) dV$$

1.9 Coordinate Systems

1.9.1 Change of Variables for Common Coordinate Systems

Coordinates	Variables						
	x	y	z	r	θ	ρ	ϕ
Cartesian	x	y	z	$\sqrt{x^2+y^2}$	$\tan^{-1}(\frac{y}{x})$	$\sqrt{x^2+y^2+z^2}$	$\cos^{-1}(\frac{z}{\rho})$
Cylindrical	$r\cos(\theta)$	$r\sin(\theta)$	z	r	θ	$r\csc(\theta)$	$\cos^{-1}(\frac{z}{\rho})$
Spherical	$\rho\sin(\phi)\cos(\theta)$	$\rho\sin(\phi)\sin(\theta)$	$\rho\cos(\phi)$	$\rho\sin(\phi)$	θ	ρ	ϕ

1.9.2 Coordinate Dot Products

Cylindrical

	$\hat{r}$	$\hat{\phi}$	$\hat{z}$
$\hat{x}$	$\cos(\phi)$	$-\sin(\phi)$	0
$\hat{y}$	$\sin(\phi)$	$\cos(\phi)$	0
$\hat{z}$	0	0	1

Spherical

	$\hat{R}$	$\hat{\theta}$	$\hat{\phi}$
$\hat{x}$	$\sin(\theta)\cos(\phi)$	$\cos(\theta)\cos(\phi)$	$-\sin(\phi)$
$\hat{y}$	$\sin(\theta)\sin(\phi)$	$\cos(\theta)\sin(\phi)$	$\cos(\phi)$
$\hat{z}$	$\cos(\theta)$	$-\sin(\theta)$	0

1.10 Trigonometry

1.10.1 Trig Identities

Half Angle Identities

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$
$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$
$$\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)}$$

Double Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$
$$\cos(2x) = 1 - 2\sin^2(x)$$

1.10.2 Hyperbolic Trig

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$
$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$
$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

1.10.3 Hyperbolic Trig Identities

$$\sinh(-x) = -\sinh(x)$$
$$\cosh(-x) = \cosh(x)$$
$$\cosh^2(x) - \sinh^2(x) = 1$$
$$1 - \tanh^2(x) = \operatorname{sech}^2(x)$$
$$\sinh(x+y) = \sinh(x)\cosh(x) + \cosh(y)\sinh(y)$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(y)\sinh(y)$$

1.11 Introduction to Electromagnetics

1.11.1 Constitutive Relations

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

$$\vec{J} = \sigma\vec{E}$$

1.11.2 Constants

Symbol	Name	Unit
ϵ	Electric Permittivity	F/m
μ	Magnetic Permeability	H/m
σ	Electrical Conductivity	S/m

1.11.3 Units

Symbol	Name	Equivalent Units
F	Farad	
H	Henri	
S	Siemens	$\frac{1}{R}$
R	Resistance	

1.12 Maxwell Equations

Integral Form	Differential Form
$\oint_S D \cdot ds = Q_{encl}$	$\nabla \cdot D = \rho_v$
$\oint_S B \cdot ds = 0$	$\nabla \cdot B = 0$
$\oint_C E \cdot dl = 0$	$\nabla \times E = 0$
$\oint_C H \cdot dl = I_{enc}$	$\nabla \times H = J$

Chapter 2

Electrostatics

2.1 Charge Densities

2.1.1 Volume

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv}$$

2.1.2 Surface

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds}$$

2.1.3 Line

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}$$

2.2 Current Density

$$\vec{J} = \rho_v \vec{u}$$

$$\vec{u} = \text{velocity of charges}$$

2.2.1 Current

$$I = \int_S \vec{J} \cdot ds$$

2.3 Resistivity and Conductivity

$$\rho = \frac{1}{\sigma}$$

Where

- ρ is the Resistivity

- σ is the Conductivity

2.3.1 Resistivity

$$R = \rho \frac{l}{A} = \frac{l}{A\sigma}$$

2.4 Electric Field and Coulomb's Law

2.4.1 Electrostatic Force

$$\vec{F} = q\vec{E}$$

2.4.2 Electric Permittivity

$$\epsilon = \epsilon_0 \epsilon_r$$

2.4.3 E Field due to Point Charges

Single Isolated Charge

$$\vec{E} = \frac{q}{4\pi\epsilon R^2}\hat{R}$$

$$\vec{E} = \frac{q}{4\pi\epsilon |R|^3}\vec{R}$$

Multiple Isolated Charges

$$\vec{E} = \frac{1}{4\pi\epsilon}\sum_{i=1}^{N} \frac{q_i}{(R_i)^2}\hat{R}_i$$

2.4.4 E Field due to Charge Distributions

Volume Distribution

$$\vec{E} = \int_v d\vec{E} = \frac{1}{4\pi\epsilon}\int_v \hat{R}\frac{\rho_v}{R^2}dv$$

Surface Distribution

$$\vec{E} = \frac{1}{4\pi\epsilon}\int_s \hat{R}\frac{\rho_s}{R^2}ds$$

Line Distribution

$$\vec{E} = \frac{1}{4\pi\epsilon}\int_l \hat{R}\frac{\rho_l}{R^2}dl$$

2.4.5 E Field due to specific geometries

Finite Line Charge

$$\vec{E} = \frac{\rho_l}{4\pi\epsilon r}\left[(\sin(\alpha_1) - \sin(\alpha_2))\hat{r} - (\cos(\alpha_1) - \cos(\alpha_2))\hat{\theta}\right]$$

Where r is the shortest vector between the observation point and the line charge ($\vec{r}$ perpendicular to line charge)

α_1, α_2 are the angles drawn by lines from the observation point P to the ends of the line charge A, B when compared to the radius

Infinite Line Charge

$$\vec{E} = \frac{\rho_l}{2\pi\epsilon r}\hat{r}$$

Ring Charge

$$\vec{E} = \frac{\rho_l bh}{2\epsilon_0(b^2 + h^2)^{3/2}}\hat{z} = \frac{Qh}{4\pi\epsilon_0(b^2 + h^2)^{3/2}}\hat{z}$$

Where $Q = 2\pi b\rho_l$

Finite Circular Disk

For a circular disk with finite radius:

$$\vec{E} = \frac{\rho_s}{2\epsilon}\hat{n}\left[1 - \frac{|h|}{\sqrt{a^2 + h^2}}\right]$$

Where a is the radius of the disk

Infinite Circular Disk/Sheet

$$\vec{E} = \frac{\rho_s}{2\epsilon}\hat{n}$$

2.5 Gauss's Law

2.5.1 Differential Form

$$\nabla \cdot \mathbf{D} = \rho_v$$

2.5.2 Integral Form

$$\oint_s D \cdot ds = Q$$

Applying Divergence Theorem

$$\int_v \nabla \cdot D dv = \oint_s D \cdot ds = Q$$

2.6 Electric Potential (Voltage)

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}$$

$$V = -\int_{\infty}^{P} \vec{E} \cdot \vec{dl}$$

2.6.1 Conservative Property (Electrostatics)

$$\oint_C \vec{E} \cdot \vec{dl} = 0$$

2.6.2 Relationship between E and V

$$\vec{E} = -\nabla V$$

2.6.3 V due to Point Charges

Single Isolated Charge

$$V = \frac{q}{4\pi\epsilon R}$$

Multiple Isolated Charges

$$V = \frac{1}{4\pi\epsilon} \sum_{i=1}^{N} \frac{q_i}{(R_i)}$$

2.6.4 Electric Potential due to Charge Distributions

Volume Distribution

$$V = \frac{1}{4\pi\epsilon} \int_v \frac{\rho_v}{R} dv$$

Surface Distribution

$$V = \frac{1}{4\pi\epsilon} \int_s \frac{\rho_s}{R} ds$$

Line Distribution

$$V = \frac{1}{4\pi\epsilon} \int_l \frac{\rho_l}{R} dl$$

2.7 Electric Dipole

$$V = \frac{Qd\cos(\theta)}{4\pi\epsilon R^2}$$

$$\vec{E} = \frac{Q\vec{d}}{4\pi\epsilon_0 R^3}(2\cos\theta\hat{R} + \sin\theta\hat{\theta}) = \frac{\vec{p}}{4\pi\epsilon_0 R^3}(2\cos\theta\hat{R} + \sin\theta\hat{\theta})$$

2.7.1 Dipole Moment

$$\vec{p} = Q\vec{d}$$

2.8 Polarization Field

$$p_{total} = \sum_{i=1}^{n\delta v} p_i$$

$\vec{P}$ is the Polarization Field

$$\vec{P} = N\vec{p}$$

$$\vec{P} = \epsilon_0(\epsilon_r - 1)\vec{E} = q\vec{d}N$$

$$\vec{P} = \chi_e(\epsilon_0)\vec{E}$$

2.8.1 Electric Susceptibility

$$\chi_e = \frac{\vec{P}}{\epsilon_0\vec{E}}$$

$$\chi_e = \epsilon_r - 1$$

2.8.2 Electric Permittivity

$$\epsilon = \epsilon_0(1 + \chi_e) = \epsilon_0\epsilon_r$$

2.8.3 Electric Flux Density

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$
$$\vec{D} = \epsilon_0\epsilon_r\vec{E} = \epsilon_0(1 + \chi_e)\vec{E} = \epsilon\vec{E}$$

2.8.4 Bound Charge

$$\Delta Q_{b,S_1} = \vec{P} \cdot \Delta s$$

2.9 Poisson and Laplace Equations

2.9.1 Poisson's Equation

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

2.9.2 Laplace's Equation

If the medium contains no free charges, Poisson's equation reduces to

$$\nabla^2 V = 0$$

2.10 Work

$$W = -Q \int_{P_1}^{P_2} \overrightarrow{E(r)} \cdot \vec{dl} = QV_{21}$$

2.10.1 Conservative Property

$$W = -Q \int_{P_1}^{P_2} \overrightarrow{E(r)} \cdot \vec{dl} = \iint_s (\nabla \times \overrightarrow{E(r)}) \cdot \vec{ds} = 0$$

2.11 Energy

2.11.1 Energy in a Discrete Charge Distribution

$$W_E = \frac{1}{2} \sum_{n=1}^{N} Q_n V_n$$

2.11.2 Energy in a Continuous Charge Distribution

$$W_E = \frac{1}{2} \iiint_v \rho_v(r) V(r) dv$$

$$W_E = \frac{1}{2} \iiint_v [\nabla \cdot V\vec{D}] dv$$

$$W_E = \frac{1}{2} \iiint_v [D\vec{(r)} \cdot E\vec{(r)}] dv$$

$$W_E = \frac{1}{2} \iiint_v \epsilon |E|^2 dv$$

2.11.3 Energy Density

$$\frac{dW_E}{dv} = \frac{1}{2} D(r) \cdot E(r) = \frac{1}{2} \epsilon |E(r)|^2$$

2.12 Capacitance

$$Q = CV$$

2.12.1 Energy

$$W_E = \frac{1}{2} CV^2$$

2.13 Electric Conductors

Perfect Conductors have infinite conductivity:

$$\sigma = \infty$$

In a conductor,

$$\vec{E} = 0$$
$$\vec{D} = 0$$

2.14 Electric Boundary Conditions

2.14.1 Tangential Condition

$$E_{1t} = E_{2t}$$

$$\hat{n} \times \vec{E}_1 = \hat{n} \times \vec{E}_2$$

Where n is the normal vector to the boundary

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

2.14.2 Normal Condition

$$D_{1n} - D_{2n} = \rho_s$$

$$\hat{n} \cdot \vec{D}_1 - \hat{n} \cdot \vec{D}_2 = \rho_s$$

Where n is the normal vector to the boundary

$$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$$

2.14.3 Dielectric Conductor Boundary

$$E_{1t} = D_{1t} = 0$$

$$D_{1n} = \epsilon_1 E_{1n} = \rho_s$$

$$\vec{D}_1 = \epsilon_1 \vec{E}_1 = \hat{n}\rho_s$$

Chapter 3

Magnetostatics

3.1 Current Density

$$\vec{J} = \rho_v \vec{u}$$

$$\vec{u} = \text{velocity of charges}$$

3.1.1 Current

$$I = \int_S \vec{J} \cdot ds$$

3.2 Magnetic Force

3.2.1 Moving Charges

$$\vec{F_m} = q\vec{u} \times \vec{B}$$

$$F_m = quB\sin(\theta)$$

Where θ is the angle between $\vec{u}$ and $\vec{B}$

3.2.2 Lorentz Force

Consider a region with both electric ($\vec{E}$) and magnetic ($\vec{B}$) fields

$$\vec{F} = \vec{F_e} + \vec{F_m} = q\vec{E} + q\vec{u} \times \vec{B} = q(\vec{E} + \vec{u} \times \vec{B})$$

3.2.3 Linear Current-Carrying Wire

$$\vec{F_m} = I\vec{L} \times \vec{B}$$

$$F_m = ILB\sin(\theta)$$

Where L is the length of the wire

3.2.4 Generalized Current-Carrying Wire

$$F_m = I \int_C \vec{dl} \times \vec{B}$$

In a uniform magnetic field, and for a **closed** loop:

$$F_m = I \left(\oint_C \vec{dl} \right) \times \vec{B} = 0$$

3.3 Magnetic Torque

3.3.1 Torque

$$\tau = \vec{r} \times \vec{F}$$

$$\vec{T} = \vec{d} \times \vec{F}$$

Where $\vec{r}$ is the vector of the moment arm, and $\vec{F}$ is the force applied

3.3.2 Magnetic Torque of a Loop

$$T = IAB\sin(\theta)$$

3.3.3 Magnetic Moment

$$\vec{m} = \hat{n}NIA = \hat{n}n$$

3.3.4 Magnetic Torque

$$\vec{T} = \vec{m} \times \vec{B}$$

3.4 Magnetic Field and Biot-Savart Law

$$d\vec{H} = \frac{I}{4\pi} \frac{\vec{dl} \times \hat{R}}{R^2}$$

$$\vec{H} = \frac{I}{4\pi} \int_l \frac{\vec{dl} \times \hat{R}}{R^2}$$

3.4.1 H Field due to Current Distributions

Current and Current Density Relationship

$$I\vec{dl} \iff \vec{J}_s ds \iff \vec{J} dv$$

Volume Current

$$\vec{H} = \frac{1}{4\pi} \int_v \frac{\vec{J} \times \hat{R}}{R^2} dv$$

Surface Current

$$\vec{H} = \frac{1}{4\pi} \int_s \frac{\vec{J}_s \times \hat{R}}{R^2} ds$$

3.4.2 H Field due to Specific Geometries

Finite Current Carrying Line

For a current carrying line parallel to the z direction

$$\vec{H} = \frac{I}{4\pi r_0} (\sin(\alpha_1) - \sin(\alpha_2)) \hat{\phi}$$

Where r_0 is the shortest distance between the observation point and the line charge

α_1, α_2 are the angles drawn by lines from the observation point P to the ends of the line charge A, B when compared to the radius

If observation point P is halfway between A and B, and h=0,

$$\vec{H} = \frac{I}{4\pi r_0} \frac{l}{\sqrt{r_0^2 + (l/2)^2}} \hat{\phi}$$

Infinite Current Carrying Line (One-sided)

For an infinite line charge with one end point

$$\vec{H} = \frac{I}{4\pi r_0} \hat{\phi}$$

Infinite Current Carrying Line

$$\vec{H} = \frac{I}{2\pi r_0} \hat{\phi}$$

Circular Loop

For a fixed point $P(0, 0, z)$ on the axis of a loop with $r = a$

$$H = \frac{I \cos(\theta)}{4\pi(a^2 + z^2)} (2\pi a) \hat{z}$$

$$H = \frac{I a^2}{2(a^2 + z^2)^{3/2}} \hat{z}$$

If P is at the center of the loop $z = 0$,

$$H = \frac{I}{2a} \hat{z}$$

Circular Loop with N turns

$$H = \frac{NIa^2}{2(a^2+z^2)^{3/2}}\hat{z}$$

If P is at the center of the loop $z = 0$,

$$H = \frac{NI}{2a}\hat{z}$$

Solenoid

For a solenoid with $d >> a$

$$H = \begin{cases} \frac{NI}{d}\hat{z} & r < a \\ 0 & r > a \end{cases}$$

Infinite Sheet

For an infinite sheet in the x-y plane with $\vec{J_s} = J_s\hat{x}$

$$H = \begin{cases} -\frac{J_s}{2}\hat{y} & z > 0 \\ +\frac{J_s}{2}\hat{y} & z < 0 \end{cases}$$

3.5 Magnetic Dipole

$$H = \frac{m}{4\pi R^3}\left(\vec{R}2\cos(\sin) + \hat{\theta}\sin(\theta)\right)$$

3.6 Gauss's Law (for Magnetism)

3.6.1 Differential Form

$$\nabla \cdot B = 0$$

3.6.2 Integral Form

$$\oint_s \vec{B} \cdot \vec{ds} = 0$$

3.7 Ampere's Law

3.7.1 Differential Form

$$\nabla \times \vec{H} = \vec{J}$$

3.7.2 Integral Form

$$\oint_C \vec{H} \cdot \vec{dl} = I$$

Applying Stoke's Theorem

$$\int_s (\nabla \times H) \cdot \vec{ds} = \int_s \vec{J} \cdot \vec{ds}$$

3.8 Vector Magnetic Potential

$$B = \nabla \times A$$

$$A = \frac{\mu}{4\pi} \int_v \frac{\vec{J}}{R} dv$$

$$I\vec{dl} \Longleftrightarrow \vec{J}_s ds \Longleftrightarrow \vec{J} dv$$

3.8.1 Scalar Magnetic Potential

If $\nabla \times \vec{H} = 0$ in a specific region, then

$$\vec{H} = -\nabla \cdot \vec{V_m}$$

3.9 Poisson and Laplace Equation

3.9.1 Poisson's Equation

$$\nabla \times B = \mu \vec{J}$$

$$\nabla^2 A = -\mu \vec{J}$$

$$\nabla \times (\nabla \times A) = \mu J$$

3.10 Magnetic Flux

$$\psi = \int_s \vec{B} \cdot \vec{ds}$$

$$\psi = \int_s (\nabla \times A) \cdot \vec{ds} = \oint_c (\vec{A} \cdot \vec{dl})$$

3.10.1 Flux Linkage

$$\lambda = N\psi$$

Where N is the number of turns

3.11 Work

$$dW = \vec{F_m} \cdot \vec{dl} = (\vec{F_m} \cdot \vec{u})dt = 0$$

$$W = \int P dt = \int IV dt$$

3.12 Energy

$$W_m = \frac{1}{2} \iiint_v |\nabla \cdot A\vec{H}| dv$$

$$W_m = \frac{1}{2} \iiint_v |B\vec{(r)} \cdot H\vec{(r)}| dv$$

$$W_m = \frac{1}{2} \iiint_v \mu |H|^2 dv$$

3.12.1 Energy Density

$$w_m = \frac{W_m}{v} = \frac{1}{2}\mu H^2$$

Expression valid for any medium with magnetic field H

3.13 Magnetic Material Properties

3.13.1 Diamagnetic

Atoms do not react to external $\vec{H}$ fields. Atoms have no permanent magnetic moments.

3.13.2 Paramagnetic

Atoms have permanent magnetic dipole moments.

3.13.3 Ferromagnetic

Atoms have permanent magnetic dipole moments.

3.13.4 Magnetic Moments

For an electron with charge $-e$ moving at constant speed u in circular orbit of radius r around a proton:

$$T = \text{period} = \frac{2\pi r}{u}$$

The circular motion of the electron constitutes a tiny loop with current I:

$$I = \frac{-e}{T} = \frac{-eu}{2\pi r}$$

Reduced Planck's Constant

$$\hbar = \frac{h}{2\pi}$$

where h is Planck's constant.

Orbital Magnetic Moment

$$m_0 = IA = \left(-\frac{eu}{2\pi r}(\pi r^2)\right) = \frac{-eur}{2} = -\left(\frac{e}{2m_e}\right) L_e$$

The smallest nonzero magnitude of m_0 occurs at $1\hbar$:

$$m_0 = -\left(\frac{e}{2m_e}\right)\hbar$$

Spin Magnetic Moment

$$m_s = -\frac{e\hbar}{2m_e}$$

3.14 Magnetization Field

$$\vec{M} = N_e \vec{m_s}$$

Where N_e is the number of electrons (total) per volume, and $\vec{m_s}$ is the Spin Magnetic Moment

$$\vec{M} = N\vec{m}$$

Where N is the number of atoms per volume, and $\vec{m}$ is the average dipole moment
$\vec{M}$ is the Magnetization Field, and m_s is the magnitude of the spin magnetic moment of a single electron in the direction of mean magnetization

$$N_e = n_e N_{atoms}$$

Where n_e is the number of electrons per atom

$$\vec{M} = \chi_m \vec{H}$$

3.14.1 Magnetic Susceptibility

$$\chi_m = \frac{\vec{M}}{\vec{H}}$$

$$\chi_m = \mu_r - 1$$

3.14.2 Magnetic Permeability

$$\mu = \mu_0(1 + \chi_m) = \mu_0\mu_r$$

3.14.3 Magnetic Flux Density

$$\vec{B} = \mu_0\vec{H} + \mu_0\vec{M}$$

3.14.4 Sample values for Magnetic Susceptibility/Permeability

Dia-Magnetic Materials

$$\chi_m = 10^{-5}$$
$$\mu_r = 1 + 10^{-5} \approx 1$$

Para-Magnetic Materials

$$\chi_m = 10^{-5}$$
$$\mu_r = 1 + 10^{-5} \approx 1$$

Ferro-Magnetic Materials

$$\chi_m = 2 \cdot 10^5$$
$$\mu_r \approx 2 \cdot 10^5$$

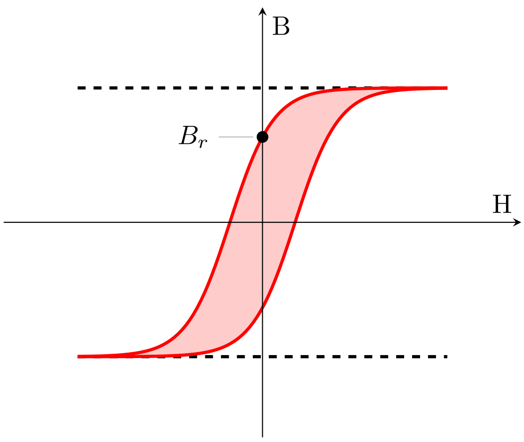
3.15 Magnetic Hysteresis (Ferromagnetic Materials)

3.15.1 Magnetic Domains

Microscopic regions where the magnetic moments of all atoms are permanently aligned, with approximate sizes of

$$\approx 10^{-10}\,\mathrm{m}^3$$

3.15.2 Magnetic Hysteresis



Where B_r is the residual magnetization

3.16 Inductance

3.16.1 Solenoid

For a long solenoid with $l/a >> 1$,

$$\vec{B} \approx \mu n I = \mu \frac{NI}{l}$$

$$\psi = \int_s \vec{B} \cdot \vec{ds} = \mu \frac{NI}{l} s$$

3.16.2 Self-Inductance

The flux linking a multi loop solenoid is

$$\lambda = \mu \frac{N^2 I}{l} s = N \cdot \psi = N \cdot \mu \frac{NI}{l} s$$

$$L = \frac{\lambda}{I} = \mu \frac{N^2}{l} s$$

Where λ is the **flux linkage**, and L is the inductance
For a general geometry, inductance L is given by

$$L = \frac{1}{I} \int_s \vec{B} \cdot \vec{ds}$$

3.16.3 Mutual Inductance

$$L_{12} = \frac{\lambda_{12}}{I_1} = \frac{N_2}{I_1} \int_{s_2} \vec{B_1} \cdot \vec{ds}$$

Where L_{12} is the mutual inductance in conductor s_2 with turns N_2 due to current I_1 flowing in conductor s_1 with turns N_1.

3.16.4 Energy

$$W_m = \frac{1}{2} L I^2$$

3.17 Magnetic Boundary Conditions

Let

- $\vec{n}$ be the normal to the Amperean loop
- $\vec{n_2}$ be the normal to the boundary

3.17.1 Tangential Condition

$$\hat{n_2} \times (\vec{H_1} - \vec{H_2}) = \vec{J_s}$$

Interface between media with Finite Conductivities

$$H_{1t} = H_{2t}$$

$$\frac{B_{2t}}{\mu_2} = \frac{B_{1t}}{\mu_1}$$

3.17.2 Normal Condition

$$B_{1n} = B_{2n}$$

$$B_1 \cdot \hat{n} = B_2 \cdot \hat{n}$$

$$\mu_1 H_{1n} = \mu_2 H_{2n}$$

Chapter 4

Electromagnetics

4.1 Maxwell Equations (General Form)

Integral Form	Differential Form
$\oint_S D \cdot ds = \int_v \rho dv$	$\nabla \cdot D = \rho_v$
$\oint_C E \cdot dl = -\frac{d}{dt} \int_S B \cdot ds$	$\nabla \times E = -\frac{\partial B}{\partial t}$
$\oint_S B \cdot ds = 0$	$\nabla \cdot B = 0$
$\oint_C H \cdot dl = \int_S \frac{\partial D}{\partial t} \cdot ds + I_{enc}$	$\nabla \times H = J + \frac{\partial D}{\partial t}$

4.2 Faraday's Law

4.2.1 Flux

$$\phi = \int_s \vec{B} \cdot \vec{ds}$$

4.2.2 Faraday's Law

$$V_{emf} = -N\frac{d\phi}{dt} = -N\frac{d}{dt}\int_s \vec{B} \cdot \vec{ds}$$

Where V_{emf} is the electromotive force produced by electromagnetic induction

$$-N\frac{d}{dt}\int_s \vec{B} \cdot \vec{ds} = \oint_C \vec{E} \cdot \vec{dl}$$

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$$

4.2.3 Conditions for EMF

Constant B Field, Constant Area

$$V_{emf} = -N\frac{d}{dt}\int_s \vec{B} \cdot \vec{ds} = 0$$

Time-Varying B Field, Constant Area

$$V_{emf} = -N\frac{d}{dt}\int_s \vec{B}\cdot\vec{ds} = V_{emf}^{tr}$$

$$V_{emf}^{tr} = \text{transformer emf}$$

$$V_{emf}^{tr} = \oint_C \vec{E}\cdot\vec{dl}$$

Constant B Field, Time-Varying Area

$$V_{emf} = -N\frac{d}{dt}\int_s \vec{B}\cdot\vec{ds} = V_{emf}^{m}$$

4.2.4 Lenz's Law

The polarity of V_{emf}^{tr} and the direction of I is always in a direction that opposes the change of magnetic flux $\phi(t)$ that produced I.

B_{ind} serves to oppose the change in $B(t)$ and not necessarily $B(t)$ itself.

4.3 Transformers

$$V_1 = -N_1\frac{d\phi}{dt} \quad V_2 = -N_2\frac{d\phi}{dt}$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

4.3.1 Impedance Matching

$$Z_{in} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$

4.4 Motional EMF

$$\vec{F_m} = q(\vec{u}\times\vec{B})$$

$$V_{emf}^{m} = \oint_C (\vec{u}\times\vec{B})\cdot dl$$

4.4.1 Generator

$$V_{emf}^{m} = A\omega B_0\sin(\alpha) = A\omega B_0\sin(\alpha)$$

$$\alpha = \omega t + C_0$$

4.5 Displacement Current

Conduction Current is current induced by moving charges. Displacement Current is induced by non-zero charge leaving/entering a given region (in other words, the time-derivative of the D field)

$$I_d = \int_S \vec{J}_d \cdot ds = \int_S \frac{\partial D}{\partial t} \cdot ds$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

4.5.1 Current Density's

$$\vec{J}_c = \sigma \vec{E}$$

$$\vec{J}_d = \frac{\partial D}{\partial t}$$

4.5.2 Ampere's Law

$$\nabla \times \vec{H} = \vec{J} + \vec{J}_D = \vec{J} + \frac{\partial D}{\partial t}$$

$$\oint_C \vec{H} \cdot \vec{dl} = \int_S \vec{J} \cdot ds + \int_S \frac{\partial D}{\partial t} \cdot \vec{ds} = I_C + I_D$$

4.5.3 Capacitor Current

$$I_d = C\frac{dV}{dt}$$

For,

$$\vec{E}(t) = \frac{v(t)}{d}\hat{y} = \frac{v_0}{d}\cos(\omega t)\hat{y}$$

Displacement current is given by,

$$I_d = -\frac{\epsilon A \omega V_0}{d}\sin(\omega t) = -CV_0\omega\sin(\omega t)$$

4.6 Traveling Waves

$$y(x,t) = A\cos(\phi) = A\cos(\phi(x,t))$$

Where

- A is the amplitude
- ϕ is the phase
- T is the period
- λ is the wavelength

- ϕ_0 is the reference phase

$$y(x,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda} + \phi_0\right)$$

$$y(x,t) = A\cos(2\pi ft - \frac{2\pi}{\lambda}x) = A\cos(\omega t - \beta x)$$

Where

- $\omega = 2\pi f$ is the Angular Frequency
- $\beta = \frac{2\pi}{\lambda}$ is the Phase Constant (or Wave Number)

4.6.1 Phase/Propagation Velocity

$$u_p = \frac{\lambda}{T}$$

$$u_p = \frac{\omega}{\beta}$$

4.6.2 Transmission Lines

For a transmission line with per unit resistance R, per unit conductance G, per unit inductance L and per unit capacitance C,

Wave Equation

$$-\frac{\partial v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t}$$

$$-\frac{\partial i(z,t)}{\partial z} = Gv(z,t) + C\frac{\partial v(z,t)}{\partial t}$$

$$-\frac{\partial}{\partial z}\hat{V}(z) = (R + j\omega L)\hat{I}(z)$$

$$-\frac{\partial}{\partial z}\hat{I}(z) = (G + j\omega C)\hat{V}(z)$$

Propagation Constant

$$\gamma = (R + j\omega L)(G + j\omega C)$$

$$\gamma = \alpha + j\beta$$

Characteristic Impedance

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Solution to Wave Equation

$$I_0^+ = \frac{\gamma}{R + j\omega L} V_0^+ \qquad I_0^- = \frac{-\gamma}{j\omega L} V_0^-$$

$$V_0^+ = |V_0^+| e^{j\hat{\phi}^+} \qquad V_0^- = |V_0^-| e^{j\hat{\phi}^-}$$

$$\hat{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

$$v(z,t) = |V_0^+| e^{-\alpha z} \cos(\phi^+ - \beta z + \omega t) + |V_0^-| e^{\alpha z} \cos(\phi^- + \beta z + \omega t)$$

Chapter 5

Electrostatic and Magnetostatic Parallels

5.1 Constants

Quality	Value	Units
Electric Permittivity	$\epsilon_0 = 8.854 \cdot 10^{-12}$	F/m
Magnetic Permeability	$\mu_0 = 4\pi \cdot 10^{-7} = 1.256 \cdot 10^{-6}$	H/m

5.2 Values

Quality	Electrostatics	Magnetostatics
Force	$\vec{F_e}$	$\vec{F_m}$
Field	$\vec{E}$	$\vec{H}$
Flux Density	$\vec{D}$	$\vec{B}$
Material Dependency	ϵ	μ
Potential	V	$\vec{A}$
Material Fields	$\vec{P}$ (Polarization)	$\vec{M}$ (Magnetization)

5.3 Equations

Quality	Electrostatics	Magnetostatics
Potential	$\vec{E} = -\nabla V$	$\vec{B} = \nabla \times \vec{A}$
Capacitance/Inductance	$C = \frac{Q}{V}$	$L = \frac{I}{\lambda}$
Energy	$W_e = \frac{1}{2}CV^2$	$W_m = \frac{1}{2}LI^2$
Gauss's Law	$\nabla \cdot D = \rho_v$	$\nabla \cdot B = 0$
Gauss's Law	$\oint_s \vec{D} \cdot \vec{ds} = Q$	$\oint_s \vec{B} \cdot \vec{ds} = 0$